EMBEDDING OF EXACT C*-ALGEBRAS AND CONTINUOUS FIELDS IN THE CUNTZ ALGEBRA $\mathcal{O}_2$

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Dedicated to Edward Effros on his 60th birthday.

Abstract. We prove that any separable exact C*-algebra is isomorphic to a subalgebra of the Cuntz algebra $\mathcal{O}_2$. We further prove that if A is a simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$, and if, in addition, A is purely infinite, then $\mathcal{O}_\infty \otimes A \cong A$.

The embedding of exact C*-algebras in $\mathcal{O}_2$ is continuous in the following sense. If A is a continuous field of C*-algebras over a compact manifold or finite CW complex X with fiber $A(x)$ over $x \in X$, such that the algebra of continuous sections of A is separable and exact, then there is a family of injective homomorphisms $\varphi_x : A(x) \to \mathcal{O}_2$ such that for every continuous section a of A the function $x \mapsto \varphi_x(a(x))$ is continuous. Moreover, one can say something about the modulus of continuity of the functions $x \mapsto \varphi_x(a(x))$ in terms of the structure of the continuous field. In particular, we show that the continuous field $\theta \mapsto A_\theta$ of rotation algebras posesses unital embeddings φ_θ in $\mathcal{O}_2$ such that the standard generators $u(\theta)$ and $v(\theta)$ satisfy

$$\max(\|\varphi_{\theta_1}(u(\theta_1)) - \varphi_{\theta_2}(u(\theta_2))\|, \|\varphi_{\theta_1}(v(\theta_1)) - \varphi_{\theta_2}(v(\theta_2))\|) < C|\theta_1 - \theta_2|^{1/2}$$

for some constant C.

0. Introduction

It has recently become clear that the exact C*-algebras form an important class of C*-algebras more general than the nuclear C*-algebras. (A C*-algebra A is called *exact* if the functor $A \otimes_{\min} -$ preserves short exact sequences.) For example, every C*-subalgebra of a nuclear C*-algebra is exact. The class of exact C*-algebras has a number of good functorial properties (see Section 7 of [Kr4]); in particular, unlike the class of nuclear C*-algebras, it is closed under passage to subalgebras. (Unfortunately, though, it is not closed under arbitrary extensions, only under "locally liftable" ones. See [Kr2] and Section 7 of [Kr4].) The reduced C*-algebras of some discrete groups (including free groups), and perhaps all discrete groups, are exact. (See Remark 7.8 of [Kr2].) Separable exact C*-algebras can be characterized as exactly those C*-algebras which occur as subquotients of the (nuclear) CAR (or 2^∞ UHF) algebra ([Kr3]).

In this paper, we show that every separable exact C*-algebra is isomorphic to a subalgebra of the Cuntz algebra $\mathcal{O}_2$. Thus, a separable C*-algebra is exact if and only if it is isomorphic to a subalgebra of of a nuclear C*-algebra, if and only if it is isomorphic to a subalgebra of the particular nuclear C*-algebra $\mathcal{O}_2$.

The methods used to prove the embedding in $\mathcal{O}_2$ show that separable exact C*-algebras which are "close" in a certain sense have nearby embeddings in $\mathcal{O}_2$. We prove that if X is a compact metric space which is sufficiently nice (certainly including all compact manifolds and all finite CW complexes), and if A is a continuous field over X in the sense of Dixmier (Chapter 10 of [Dx]) such that the algebra of continuous sections of A is separable and exact, than A has a continuous representation in $\mathcal{O}_2$. That is, there is a collection of injective homomorphisms φ_x from the fibers $A(x)$ of A to $\mathcal{O}_2$ such that, for every continuous section a of A, the function $x \mapsto \varphi_x(a(x))$ is continuous. Moreover, one can say something about the "smoothness" of these functions. For example, we show that there are injective homomorphisms φ_θ from the rotation algebras A_θ (rational and irrational) to $\mathcal{O}_2$

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such that, if $u(\theta)$ and $v(\theta)$ denote the standard generators of A_θ, then there is a constant C such that

$$\max(\|\varphi_{\theta_1}(u(\theta_1)) - \varphi_{\theta_2}(u(\theta_2))\|, \|\varphi_{\theta_1}(v(\theta_1)) - \varphi_{\theta_2}(v(\theta_2))\|) < C|\theta_1 - \theta_2|^{1/2}$$

for all $\theta_1, \theta_2 \in \mathbb{R}$. Haagerup and Rørdam [HR] obtained representations on a Hilbert space with these properties, and showed that even there the exponent $\frac{1}{2}$ can't be improved. Our work provides an independent proof of the Haagerup and Rørdam representation–a proof not using unbounded operators.

Blanchard [Bl] has a somewhat different approach to the representation of continuous fields in $\mathcal{O}_2$. He is able to allow more general base spaces X, but obtains no information on smoothness. In particular, his methods do not provide the $\mathrm{Lip}^{1/2}$ representation (or a Lip^{α} representation for any α) of the field of rotation algebras in $\mathcal{O}_2$.

In [El], George Elliott initiated a program for the classification of separable nuclear simple C*-algebras of real rank zero in terms of K-theoretic invariants. The purely infinite case was first tackled in [Rr1] (building on the work of [BKRS]), and has since been investigated in a number of papers. (See [BEEK], [ElR], [Ln2], [Ln3], [LP1], [LP2], [Rr1], [Rr2], [Rr3], and [Rr4].) The classification program predicts that if A is separable, nuclear, unital, and simple, then $\mathcal{O}_2 \otimes A$ should be isomorphic to $\mathcal{O}_2$, and that if, in addition, A is purely infinite, then $\mathcal{O}_\infty \otimes A$ should be isomorphic to A. (The Künneth formula for C*-algebras [Sc] implies that $K_*(\mathcal{O}_2 \otimes A) \cong K_*(\mathcal{O}_2)$ and $K_*(\mathcal{O}_\infty \otimes A) \cong K_*(A)$.) We use the result on embeddings in $\mathcal{O}_2$, together with some of the lemmas in its proof and some additional results, to prove that these isomorphisms do in fact hold: $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ for separable nuclear unital simple A, and $\mathcal{O}_\infty \otimes A \cong A$ for separable, nuclear, unital, purely infinite, and simple A.

These results are the starting point for a proof by the second author of a general classification theorem for separable nuclear unital purely infinite simple C*-algebras satisfying the universal coefficient theorem [Ph2]. The first author also has an independent proof [Kr5] of this classification theorem, which does not use the isomorphisms of tensor products above directly, but rather obtains them as a consequence of the general classification result.

This paper is organized as follows. The rest of the introduction contains some general terminology and notation, and some more or less well known results that we use often enough that it is convenient to have them restated here. In Section 1 we prove several technical results on approximation and perturbation of unital completely positive maps. In the second section, we use these results to prove that separable exact C*-algebras can be embedded in $\mathcal{O}_2$. Section 3 contains the proofs of $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ and $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$. Those interested only in the classification program need read no further. In the fourth section, we present some preliminaries on continuous fields, including results based on earlier papers and results based on Section 1 of this paper. In the fifth section we give a version of the argument Haagerup and Rørdam use in [HR] to obtain continuous representations of continuous fields when one merely knows that any two nearby fibers have nearby embeddings. Section 6 is devoted to the detailed examination of the field of rotation algebras.

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Here is some general notation we use throughout. M_n is the algebra of $n \times n$ matrices, with matrix units $\{e_{ij}\}$. If H is a separable infinite dimensional Hilbert space, then $K = K(H)$ denotes the algebra of compact operators on H, and $L(H)$ denotes the algebra of bounded operators on H. The unitization of a C*-algebra A is denoted $A^\dagger$; this means we add a new unit even if A already has one. We let $\tilde{A}$ denote A if A is unital and $A^\dagger$ if not. The unitary group of a unital C*-algebra A is denoted $U(A)$, and $U_0(A)$ is the connected component of $U(A)$ containing the identity. When we refer to a unital subalgebra B of a C*-algebra A, we implicitly mean that the B is supposed to contain the identity of A.

We present here some (well) known results which are used frequently enough that is is convenient to restate them.

The first part of the inequality in the following estimate is the best possible, as can be seen by taking $p = 1$ and x to be a positive real scalar less than 1. Note that $1 - (1 - \delta)^{1/2}$ is approximately $\delta/2$ for small δ. The second part of the inequality gives the best linear estimate over the relevant range, as can be seen by letting $\|x^*x - p\| \to 1$ (for example, letting $x \to 0$).

Lemma 0.1. Let A be a C*-algebra, let $p \in A$ be a projection, and let $x \in A$ satisfy $xp = x$ and $\|x^*x - p\| < 1$. Then the formula $v = x(x^*x)^{-1/2}$ (functional calculus evaluated in pAp) defines a partial isometry in A with $v^*v = p$. Moreover,

$$\|v - x\| \leq 1 - (1 - \|x^*x - p\|)^{1/2} \leq \|x^*x - p\|.$$

Proof: It is well known that if $\|x^*x - p\| < 1$ then v is a partial isometry satisfying $v^*v = p$. For the rest, let $\delta = \|x^*x - p\|$. A calculation, using the fact that $(x^*x)^{1/2} \in pAp$, shows that

$$(v - x)^*(v - x) = [(x^*x)^{1/2} - p]^2$$

and

$$\|v - x\| = \|(x^*x)^{1/2} - p\| \leq \sup\{|t^{1/2} - 1| : |t - 1| \leq \delta\} = 1 - (1 - \delta)^{1/2}.$$

This gives the first inequality. For the second, note that $(1 - \delta)^{1/2} \geq 1 - \delta$, whence $(1 - \delta)^{1/2} - 1 \geq -\delta$. ∎

Theorem 0.2. (Theorem 3.1 of [CE1]; also see [Kr1].) A separable unital C*-algebra A is nuclear if and only if there are sequences of unital completely positive maps $S_k : A \to M_{n(k)}$ and $S_k : M_{n(k)} \to A$, for suitable $n(k)$, such that $\lim_{k\to\infty} T_k \circ S_k(a) = a$ for all $a \in A$.

The maps in [CE1] are not required to be unital, but this is easily fixed. See the note on this point in the proof of Proposition 4.3 of [EH].

Theorem 0.3. (Choi-Effros Lifting Theorem; see the corollary to Theorem 7 of [Ar].) Let A be a separable nuclear unital C*-algebra, let B be a unital C*-algebra, and let J be an ideal of B, with quotient map $\pi : B \to B/J$. Then for every unital completely positive map $S : A \to B/J$, there is a unital completely positive map $T : A \to B$ which lifts S, that is, such that $\pi \circ T = S$.

Proposition 0.4. Let A and B be unital C*-algebras. Let $E \subset A$ be an operator system (in the sense of Choi and Effros; see [CE2]), and let $S : E \to B$ be a nuclear unital completely positive map. Then for every finite dimensional operator system $E_0 \subset E$ and every $\varepsilon > 0$, there is a unital completely positive map $T : A \to B$ such that $\|T|_{E_0} - S|_{E_0}\| < \varepsilon$.

Proof: This is similar to Proposition 4.3 of [EH]. Since S is nuclear, there are n and unital completely positive maps $P_0 : E \to M_n$ and $Q : M_n \to B$ such that $\|Q \circ P_0|_{E_0} - S|_{E_0}\| < \varepsilon$. The Arveson extension theorem (Theorem 6.5 of [Pl]) provides a unital completely positive map $P : A \to M_n$ such that $P|_{E_0} = P_0|_{E_0}$. Set $T = Q \circ P$. ∎

Definition 0.5. Let A and B be C*-algebras, with A separable and B unital. Two homomorphisms $\varphi, \psi : A \to B$ are *approximately unitarily equivalent* if there is a sequence (u_n) of unitaries in B such that $\lim_{n\to\infty} \|u_n\varphi(a)u_n^* - \psi(a)\| = 0$ for all $a \in A$.

We will frequently use the following special case of Elliott's approximate intertwining argument, of which the original form is Theorem 2.1 of [El].

Lemma 0.6. Let A and B be separable unital C*-algebras, and let $\varphi : A \to B$ and $\psi : B \to A$ be homomorphisms such that $\psi \circ \varphi$ is approximately unitarily equivalent to id_A and $\varphi \circ \psi$ is approximately unitarily equivalent to id_B . Then $A \cong B$.

Proof: This is contained in the proof of Theorem 6.2 (1) of [Rr2]. (Or see Proposition A of [Rr3].) ∎

Proposition 0.7. Let D be a unital purely infinite simple C*-algebra. Then any two unital homomorphisms from $\mathcal{O}_2$ to D are approximately unitarily equivalent.

Proof: This is a special case of Theorem 3.6 of [Rr1]. The required two conditions on D (that $U(D)/U_0(D) \to K_1(D)$ be an isomorphism and that D have finite exponential length in the sense of [Rn]) follow from Theorem 1.9 of [Cn2] and from [Ph1] or [Ln1] respectively. ∎

Theorem 0.8. ([Rr3]) $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$.

1. Completely positive maps and perturbation

In this section we prove several approximation and perturbation results which are frequently required later in the paper. At the end of the section, we combine these results (without using their full strength) to prove a lemma on approximate unitary equivalence of homomorphisms to tensor products with $\mathcal{O}_2$. Combined with the isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$, it implies that two unital injective homomorphisms from a separable exact C*-algebra to $\mathcal{O}_2$ are approximately unitarily equivalent.

The first of the three main results of this section is Proposition 1.7, which shows that a nuclear unital completely positive map from a unital purely infinite simple C*-algebra to itself can be approximated on finite sets by maps of the form $a \mapsto s^* a s$ for isometries s. The proof is somewhat different from that of a similar result in [Kr5], relying much more heavily on properties of purely infinite simple C*-algebras. The next important result is Lemma 1.10, in which we show, under suitable exactness and nuclearity assumptions, that if $S, T : A \to B$ are unital and completely positive and S is sufficiently close to being completely isometric on a finite dimensional operator system E, then $T \circ S^{-1}$ can be approximated on $S(E)$ by a unital completely positive map. The last main result is Lemma 1.12, in which approximate "similarity" via isometries in A is shown to imply approximate unitary equivalence in $\mathcal{O}_2 \otimes A$. We use it to show that two injective unital homomorphisms from a separable unital exact C*-algebra to $\mathcal{O}_2$ are approximately unitarily equivalent.

We begin with some preliminary lemmas on purely infinite simple C*-algebras.

Lemma 1.1. Let A be a C*-algebra, let $a, h \in A$ be selfadjoint elements with $0 \leq a \leq 1$ and $0 \leq h \leq 1$, and let $q \in A$ be a projection. Then $\|qa - q\| \leq 12 \|qhah - q\|^{1/3}$.

Proof: Represent A faithfully on a Hilbert space H. Note that $\|qa-q\| = \|aq-q\|$ and $\|qhah-q\| = \|hahq-q\|$. It suffices to show that if $\xi \in qH$, then $\|a\xi - \xi\| \leq 12 \|hah\xi - \xi\|^{1/3}$.

We first claim that if $b \in L(H)$ satisfies $0 \leq b \leq 1$ and $\eta \in H$ satisfies $\|\eta\| = 1$, then $\|b\eta - \eta\| \leq 4(1 - \|b\eta\|)^{1/3}$. To see this, set $\delta = 1 - \|b\eta\|$, let $\rho = \delta^{1/3}$, and let $p \in L(H)$ be the spectral projection for b corresponding to $[1-\rho, 1]$. Then

$$\begin{aligned} (1-\delta)^2 &= \|b\eta\|^2 = \|bp\eta\|^2 + \|b(1-p)\eta\|^2 \leq \|p\eta\|^2 + (1-\rho)^2 \|(1-p)\eta\|^2 \\ &= 1 - \|(1-p)\eta\|^2 + (1-\rho)^2 \|(1-p)\eta\|^2 = 1 - \rho(2-\rho)\|(1-p)\eta\|^2. \end{aligned}$$

It follows that

$$\|(1-p)\eta\| \leq \sqrt{\left(\frac{2-\delta}{2-\rho}\right)\left(\frac{\delta}{\rho}\right)} \leq \sqrt{2}\delta^{1/3}.$$

So

$$\|b\eta - \eta\| \leq \|b\| \|(1-p)\eta\| + \|bp\eta - p\eta\| + \|(1-p)\eta\| \leq 2\sqrt{2}\delta^{1/3} + \rho < 4\delta^{1/3}.$$

This proves the claim.

Now let $\xi \in qH$ satisfy $\|\xi\| = 1$. Since $\|a\|, \|h\| \leq 1$, we have $\|h\xi\| \geq 1 - \|hah\xi - \xi\|$. Applying the claim to h and ξ, we get $\|h\xi - \xi\| \leq 4 \|hah\xi - \xi\|^{1/3}$. Also, with $\eta = \frac{1}{\|h\xi\|} h\xi$, we have

$$\|a\eta\| \geq \frac{1}{\|h\xi\|} \|hah\xi\| \geq \frac{1}{\|h\xi\|}(1 - \|hah\xi - \xi\|) \geq 1 - \|hah\xi - \xi\|,$$

so

$$\|ah\xi - h\xi\| = \|h\xi\| \|a\eta - \eta\| \leq \|h\xi\| \cdot 4 \|hah\xi - \xi\|^{1/3} \leq 4 \|hah\xi - \xi\|^{1/3}.$$

Therefore

$$\|a\xi - \xi\| \leq \|a\| \|\xi - h\xi\| + \|ah\xi - h\xi\| + \|h\xi - \xi\| \leq 12 \|hah\xi - \xi\|^{1/3}.$$

∎

Lemma 1.2. Let A be a purely infinite simple C*-algebra, and let $a_1, \ldots, a_n \in A$ be positive elements with $\|a_j\| = 1$ for all j. Then for every $\varepsilon > 0$ there are nonzero mutually orthogonal projections $p_1, \ldots, p_n \in A$ such that $\|p_j a_j - p_j\| < \varepsilon$ for all j.

Proof: Choose $\delta > 0$ with $12(2\delta)^{1/3} < \varepsilon$. Choose an irreducible representation π of A on a Hilbert space H. By induction, we construct a sequence $\xi_1, \dots, \xi_n$ of orthogonal unit vectors in H with $\|\pi(a_j)\xi_j - \xi_j\| < \delta$ for $1 \leq j \leq n$. Choose ξ_1 to be any unit vector in the spectral subspace for $\pi(a_1)$ corresponding to $[1-\delta, 1]$. Given $\xi_1, \dots, \xi_j$, let H_0 be the spectral subspace for $\pi(a_{j+1})$ corresponding to $[1-\delta, 1]$. Since $\pi(A)$ contains no compact operators, this space must be infinite dimensional. Therefore it must nontrivially intersect the finite codimension subspace $\mathrm{span}(\xi_1, \dots, \xi_j)^{\perp}$, and we take ξ_{j+1} to be any unit vector in the intersection.

Let $p_j \in L(H)$ be the projection onto $\mathbb{C}\xi_j$, and let $p = p_1 + \cdots + p_n$ be the projection onto $\mathrm{span}(\xi_1, \dots, \xi_n)$. Let

$$L = \{a \in A : \pi(a)p = 0\} \quad \text{and} \quad N = \{a \in A : \pi(a)p = p\pi(a)\}.$$

Then L is a left ideal of A, N is a C*-subalgebra of A, and $L \cap L^*$ is an ideal in N. Define a unital completely positive map $T : A \to L(pH) \cong M_n$ by $T(a) = p\pi(a)p$. Then $T|_N$ is a homomorphism with kernel $L \cap L^*$. The Kadison Transitivity Theorem implies it is surjective. Indeed, let $u \in L(pH)$ be unitary. Since π is injective, Theorem 5.4.5 of [KR] provides a unitary $v \in \tilde{A}$ such that $p\pi(v)p = u$. Since $\pi(v)$ and $p\pi(v)p$ are both unitary, p must commute with $\pi(v)$. (This is well known, but see a closely related result in Lemma 1.11 below.) So $v \in \tilde{N}$ and $T(v) = u$. This shows that the image of $\tilde{N}$ contains all unitaries in $L(pH)$, and so is all of $L(pH)$. The image of N is an ideal of codimension at most 1. We may clearly assume $n \geq 2$; then $L(pH)$ has no proper ideals of codimension at most 1, so $T|_N$ must be surjective.

By Theorem 4.6 of [Lr] (essentially Proposition 2.6 of [AP1]), there are $h_1, \dots, h_n \in N$ satisfying $T(h_j) = p_j$, $0 \leq h_j \leq 1$, and $h_j h_k = 0$ for $j \neq k$. Then

$$\|h_j a_j h_j\| \geq \|p\pi(h_j)\pi(a_j)\pi(h_j)p\| = \|p_j \pi(a_j) p_j\| = \langle \pi(a_j)\xi_j, \xi_j \rangle \geq 1 - \delta.$$

The proof of Lemma 1.7 of [Cn2] provides a projection $q_j \in \overline{h_j A h_j}$ such that $\|q_j h_j a_j h_j - q_j\| < 2\delta$. Since $h_j h_k = 0$ for $j \neq k$, we also have $q_j q_k = 0$ for $j \neq k$. Moreover, $\|q_j a_j - q_j\| \leq 12(2\delta)^{1/3} < \varepsilon$ by the previous lemma. ∎

Lemma 1.3. Let A be a unital purely infinite simple C*-algebra, and let $F \subset A$ be a finite subset consisting of positive elements a satisfying $\alpha \leq a \leq \beta$ for fixed positive real numbers α and β. Then for all $\rho > 0$ there is $\delta > 0$ such that the following holds: If there are positive elements $c_1, \dots, c_n \in A$ with $\|c_j\| = 1$ such that $\|c_j a c_j - \lambda_j(a) c_j^2\| < \delta$ for $1 \leq j \leq n$, $a \in F$, and some numbers $\lambda_j(a) \in [\alpha, \beta]$, then there are nonzero mutually orthogonal projections $p_1, \dots, p_n \in A$ such that $\|p_j a p_k - \delta_{jk}\lambda_j(a) p_j\| < \rho$ for $1 \leq j \leq n$ and $a \in F$, where δ_{jk} is the Kronecker delta.

Proof: Without loss of generality $\beta = 1$, $1 \in F$, and $\lambda_j(1) = 1$ for all j. (If $\beta \neq 1$, we can rescale by multiplying by $1/\beta$. If 1 is already in F but $\lambda_j(1) \neq 1$, it does no harm to redefine $\lambda_j(1)$.) The proof is now by induction on the number of elements of F, but to make the argument work we require the additional conclusion $\|p_j c_j - p_j\| < \rho$ for $1 \leq j \leq n$. If F has only one element, then it is 1, and the existence of the required projections p_j is just Lemma 1.2.

Assume now that F has more than one element, and that the lemma is known to hold for all smaller such sets. Define $\mu = \alpha\rho/15$. (Then in particular $0 < \mu < \rho/5$.) Choose $b \in F$ with $b \neq 1$, and let $F_0 = F \setminus \{b\}$. Use the induction hypothesis to choose δ with $0 < \delta < \mu$ such that the conclusion of the lemma holds for F_0 in place of F and μ in place of ρ. Let $q_1, \dots, q_n$ be the nonzero mutually orthogonal projections provided by the conclusion.

Set $x_j = \lambda_j(b)^{-1/2} q_j b^{1/2}$. Then

$$\begin{aligned} \|x_j x_j^* - q_j\| &\leq \frac{1}{\alpha}\|q_j b q_j - \lambda_j(b) q_j\| \\ &\leq \frac{1}{\alpha}\left(4\|q_j c_j - q_j\| + \|q_j\|\|c_j b c_j - \lambda_j(b) c_j^2\|\|q_j\|\right) < \frac{1}{\alpha}(4\mu + \delta) \leq \frac{5\mu}{\alpha} = \frac{\rho}{3}. \end{aligned}$$

By Lemma 0.1 there exist partial isometries v_j such that $v_j v_j^* = q_j$ and $\|v_j - x_j\| < \rho/3$.

Lemma 1.2 provides nonzero mutually orthogonal projections $e_1, \dots, e_n$ such that $\|e_j v_j^* v_j - e_j\| < \rho/24$ for $1 \leq j \leq n$. It follows that $\|v_j^* v_j e_j v_j^* v_j - e_j\| < \rho/12$, and applying functional calculus to $v_j^* v_j e_j v_j^* v_j$ yields a

projection $f_j \leq v_j^* v_j$ such that $\|f_j - e_j\| < \rho/6$. Since the e_j are mutually orthogonal, it follows that $\|f_j f_k\| < \rho/3$ for $j \neq k$. Define $p_j = v_j f_j v_j^* \leq q_j$.

Since $p_j \leq q_j$, the p_j are mutually orthogonal, and the corresponding properties of the q_j imply that $\|p_j a p_k - \delta_{jk}\lambda_j(a)p_j\| < \mu \leq \rho$ for $1 \leq j,\, k \leq n$ and $a \in F_0$ and that $\|p_j c_j - p_j\| < \mu \leq \rho$ for $1 \leq j \leq n$. Furthermore, in the estimate of $\|x_j x_j^* - q_j\|$ above we saw that $\|q_j b q_j - \lambda_j(b) q_j\| < 5\mu$, which is at most ρ. The same estimate therefore holds with p_j in place of q_j.

It remains to estimate $\|p_j b p_k\|$ for $j \neq k$. Using $\lambda_j(b)^{1/2},\ \lambda_k(b)^{1/2} \leq 1$ in the last step, we have, for $j \neq k$,

$$\begin{aligned}
\|p_j b p_k\| &= \|p_j q_j b q_k p_k\| = \lambda_j(b)^{1/2}\lambda_k(b)^{1/2}\|p_j x_j x_k^* p_k\| \leq \lambda_j(b)^{-1/2}\lambda_k(b)^{-1/2}\|f_j v_j^* x_j x_k^* v_k f_k\| \\
&\leq \lambda_j(b)^{1/2}\lambda_k(b)^{1/2}\left(\|f_j v_j^* v_j v_k^* v_k f_k\| + \|x_k\|\|x_j - v_j\| + \|x_k - v_k\|\right) \\
&< \lambda_j(b)^{1/2}\lambda_k(b)^{1/2}\left(\|f_j f_k\| + \lambda_k(b)^{-1/2}\rho/3 + \rho/3\right) \leq \rho.
\end{aligned}$$

■

The main technical parts of the proof of the next lemma are the excision of pure states (see [AAP]) and the previous lemma. (Note that Proposition 2.3 of [AAP] and Lemma 1.4 imply that every state on a unital purely infinite simple C*-algebra is a weak* limit of pure states.)

Lemma 1.4. Let A be a unital purely infinite simple C*-algebra, and let ω be a state on A. Then for every $\varepsilon > 0$ and every finite subset $F \subset A$ there exists a nonzero projection $p \in A$ such that $\|pap - \omega(a)p\| < \varepsilon$ for all $a \in F$.

Proof: Without loss of generality, we may assume $1 \in F$ and that F consists of positive elements of norm at most 1. It further suffices to prove the lemma using the set $\{\frac{1}{2}(1+a) : a \in F\}$ instead of F; thus we may assume without loss of generality that $a \geq \frac{1}{2}$ for all $a \in F$. In particular, if μ is any state on A, then $\mu(a) \geq \frac{1}{2}$ for all $a \in F$.

Since the set of all states is the weak* closed convex hull of the set of pure states, there are $\alpha_1, \ldots, \alpha_n \in [0,1]$ and pure states $\omega_1, \ldots, \omega_n$ of A such that $\sum_{j=1}^n \alpha_j = 1$ and $|\omega(a) - \sum_{j=1}^n \alpha_j \omega_j(a)| < \varepsilon/2$ for all $a \in F$. Choose $\delta > 0$ as in Lemma 1.3 for F and the number $\rho = \varepsilon/(2n^2)$. Excision of pure states (see Proposition 2.2 of [AAP]) implies that there are positive elements $c_1, \ldots, c_n \in A$ of norm 1 such that $\|c_j a c_j - \omega_j(a) c_j^2\| < \delta$ for $1 \leq j \leq n$ and $a \in F$. Apply Lemma 1.3 with $\lambda_j = \omega_j|_F$ to obtain nonzero mutually orthogonal projections $q_1, \ldots, q_n \in A$ such that $\|q_j a q_k - \delta_{jk}\omega_j(a) q_j\| < \varepsilon/(2n^2)$.

Since a is purely infinite and simple, there exist isometries $s_1, \ldots, s_n \in A$ such that $p_j = s_j s_j^* \leq q_j$. Define $s = \sum_{j=1}^n \alpha_j^{1/2} s_j$. Since the p_j are orthogonal and $\sum_{j=1}^n \alpha_j = 1$, one immediately checks that s is an isometry. Define $p = ss^*$. Then, for $a \in F$ we have

$$\begin{aligned}
\|pap - \omega(a)p\| &= \|s^* a s - \omega(a)\cdot 1\| < \frac{\varepsilon}{2} + \sum_{j,\,k=1}^{n} \alpha_j^{1/2}\alpha_k^{1/2}\|s_j^* a s_k - \delta_{jk}\omega_j(a)\cdot 1\| \\
&= \frac{\varepsilon}{2} + \sum_{j,\,k=1}^{n} \alpha_j^{1/2}\alpha_k^{1/2}\|p_j a p_k - \delta_{jk}\omega_j(a) p_j\| < \frac{\varepsilon}{2} + n^2\frac{\varepsilon}{2n^2} = \varepsilon,
\end{aligned}$$

since $\alpha_j \leq 1$ and $p_j \leq q_j$. ■

Lemma 1.5. Let A be a unital purely infinite simple C*-algebra, let $T : A \to M_n$ be a unital completely positive map, and let $\varphi : M_n \to A$ be a (not necessarily unital) homomorphism. Then for every $\varepsilon > 0$ and every finite subset $F \subset A$ there exists a partial isometry $s \in A$ such that $s^*s = \varphi(1)$ and $\|s^* a s - \varphi(T(a))\| < \varepsilon$ for all $a \in F$.

Proof: Without loss of generality we may assume $1 \in F$ and that all elements of F have norm at most 1. Choose $\rho > 0$ such that $\rho \leq \min(1, \varepsilon/4)$, and also so small that if q and q' are projections such that $\|qq'\| < 4\rho$ then there are orthogonal projections r and r' unitarily equivalent to q and q' respectively. Define $\delta = \rho/(3n^3)$; then $0 < \delta \leq \varepsilon/n^3$.

Let $\{\xi_1, \ldots, \xi_n\}$ be the standard orthonormal basis of $\mathbb{C}^n$, and let e_{kl}, for $1 \leq k, l \leq n$, be the standard matrix units satisfying $e_{kl}\xi_j = \delta_{jl}\xi_k$. Following Theorem 5.1 of [Pl] and the preceding discussion, define a state on $M_n \otimes A$ by

$$\omega\left(\sum_{k,\, l=1}^{n} e_{kl} \otimes a_{kl}\right) = \frac{1}{n}\sum_{k,\, l=1}^{n} \langle T(a_{kl})\xi_l, \xi_k\rangle.$$

(Note that $\omega(1) = 1$ because $T(1) = 1$.) As there, we then have

$$T(a) = n\sum_{k,\, l=1}^{n} \omega(e_{kl} \otimes a)e_{kl}$$

for $a \in A$. By Lemma 1.4, there is a nonzero projection $p_0 \in M_n \otimes A$ such that $\|p_0(e_{ij} \otimes a)p_0 - \omega(e_{ij} \otimes a)p_0\| < \delta$ for all $a \in F$ and $1 \leq i, j \leq n$.

Since $M_n \otimes A$ is purely infinite and simple, there is a nonzero projection $p \in M_n \otimes A$, with $p \leq p_0$, such that there are partial isometries $s_1, \ldots, s_n \in M_n \otimes A$ satisfying $s_j s_j^* = p$ and $s_j^* s_j = e_{11} \otimes \varphi(e_{jj})$. We then have

$$\|s_i^*(e_{kl} \otimes a)s_j - \omega(e_{kl} \otimes a)(e_{11} \otimes \varphi(e_{ij}))\| < \delta$$

for all $a \in F$ and $1 \leq i, j\, k, l \leq n$.

Define $c = \sum_{k=1}^{n}(e_{1k} \otimes 1)s_k \in M_n \otimes A$. One checks that for $a \in F$ we have

$$c^*(e_{11} \otimes a)c = \sum_{k,\, l=1}^{n} s_k^*(e_{kl} \otimes a)s_l,$$

from which it follows that

$$\begin{aligned} &\|nc^*(e_{11} \otimes a)c - e_{11} \otimes \varphi(T(a))\| \\ \leq\ & n\sum_{k,\, l=1}^{n} \|s_k^*(e_{kl} \otimes a)s_l - \omega(e_{kl} \otimes a)(e_{11} \otimes \varphi(e_{kl}))\| < n^3\delta. \end{aligned}$$

Putting in particular $a = 1$, we obtain $\|nc^*(e_{11} \otimes 1)c - e_{11} \otimes \varphi(1)\| < n^3\delta$. Set

$$d = \sqrt{n}(e_{11} \otimes 1)c(e_{11} \otimes \varphi(1)) \in (e_{11} \otimes 1)(M_n \otimes A)(e_{11} \otimes 1).$$

Then $\|d^*d - e_{11} \otimes \varphi(1)\| < n^3\delta$, so Lemma 0.1 implies that $d(d^*d)^{-1/2}$ (with functional calculus taken in $(e_{11} \otimes \varphi(1))(M_n \otimes A)(e_{11} \otimes \varphi(1)))$ is a partial isometry in $(e_{11} \otimes 1)(M_n \otimes A)(e_{11} \otimes 1)$ satisfying

$$[d(d^*d)^{-1/2}]^*[d(d^*d)^{-1/2}] = e_{11} \otimes \varphi(1) \quad \text{and} \quad \|d(d^*d)^{-1/2} - d\| < n^3\delta \leq \rho.$$

Let $s \in A$ be the partial isometry such that $d(d^*d)^{-1/2} = e_{11} \otimes s$. We check that s satisfies the required estimates. First note that $\|nc^*(e_{11} \otimes a)c - e_{11} \otimes \varphi(T(a))\| < n^3\delta$ implies that

$$\|d^*(e_{11} \otimes a)d - e_{11} \otimes \varphi(T(a))\| < n^3\delta.$$

Moreover, $\|e_{11} \otimes s - d\| < \rho$ implies that $\|d\| < 1 + \rho < 2$. Therefore

$$\|s^*as - \varphi(T(a))\| = \|(e_{11} \otimes s)^*(e_{11} \otimes a)(e_{11} \otimes s) - e_{11} \otimes \varphi(T(a))\| < 2\rho + \rho + n^3\delta \leq \varepsilon,$$

since $\|a\| \leq 1$ for all $a \in F$. ∎

The following lemma and its proof were inspired by Kasparov's Stinespring theorem for Hilbert modules ([Ks]).

Lemma 1.6. Let A be a unital C*-algebra, and let $T : M_n \to A$ be unital and completely positive. Denote by $\{e_{ij}\}$ the standard system of matrix units in M_n. Then there exists a partial isometry $t \in M_n \otimes M_n \otimes A$ such that

$$t^*t = e_{11} \otimes e_{11} \otimes 1 \quad \text{and} \quad t^*(b \otimes 1 \otimes 1)t = e_{11} \otimes e_{11} \otimes T(b)$$

for $b \in M_n$.

Proof: Set $x = \sum_{i,\,j=1}^{n} e_{ij} \otimes e_{ij} \in M_n \otimes M_n$. Note that $n^{-1}x$ is a projection, so x is positive. Therefore so is

$$y = (\mathrm{id}_{M_n} \otimes T)(x) = \sum_{i,\,j=1}^{n} e_{ij} \otimes T(e_{ij}) \in M_n \otimes A.$$

Write $y^{1/2} = \sum_{i,\,j=1}^{n} e_{ij} \otimes a_{ij}$ for suitable $a_{ij} \in M_n \otimes A$. Since $y^{1/2}$ is selfadjoint and has square y, we obtain $a_{ik}^* = a_{ki}$ and $\sum_{j=1}^{n} a_{ij} a_{jk} = T(e_{ik})$ for $1 \leq i,\, k \leq n$.

Define $t = \sum_{i,\,j=1}^{n} e_{i1} \otimes e_{j1} \otimes a_{ji}$. A computation shows that

$$t^*(e_{ik} \otimes 1 \otimes 1)t = e_{11} \otimes e_{11} \otimes \sum_{j=1}^{n} a_{ij} a_{jk} = e_{11} \otimes e_{11} \otimes T(e_{ik}).$$

It follows that $t^*(b \otimes 1 \otimes 1)t = e_{11} \otimes e_{11} \otimes T(b)$ for all $b \in M_n$. In particular, $t^*t = e_{11} \otimes e_{11} \otimes T(1) = e_{11} \otimes e_{11} \otimes 1$. The last equation implies that t is a partial isometry. ∎

Proposition 1.7. Let A be a unital purely infinite simple C*-algebra, and let $V : A \to A$ be a nuclear unital completely positive map. Then for every $\varepsilon > 0$ and every finite subset $F \subset A$ there exists a nonunitary isometry $s \in A$ such that $\|s^* a s - V(a)\| < \varepsilon$ for all $a \in F$.

Proof: By the definition of nuclearity, V is a pointwise norm limit of maps of the form $T \circ S$, with $n \in \mathbb{N}$ and $S : A \to M_n$, $T : M_n \to A$ unital and completely positive. Therefore it suffices to prove the proposition for maps of the form $T \circ S$.

Let $\{e_{ij}\}$ be the standard system of matrix units in M_n. Since A is purely infinite and simple, so is $M_n \otimes M_n \otimes A$. Therefore there is $t_1 \in M_n \otimes M_n \otimes A$ such that

$$t_1^* t_1 = 1 \quad \text{and} \quad t_1 t_1^* < e_{11} \otimes e_{11} \otimes 1.$$

Apply the previous lemma to T, and call the resulting partial isometry t_2. Define a (nonunital) homomorphism $\varphi_0 : M_n \to A$ by identifying A with the corner

$$A_0 = (e_{11} \otimes e_{11} \otimes 1)(M_n \otimes M_n \otimes A)(e_{11} \otimes e_{11} \otimes 1),$$

and setting $\varphi_0(b) = t_1(b \otimes 1 \otimes 1)t_1^*$. Set $t = t_1 t_2$, which is an isometry in A_0, and regard it as an element of A. Then, using the result of the previous lemma, we have $T(b) = t^* \varphi_0(b) t$ for all $b \in M_n$.

Let $p = t_1 t_1^*$, which we also regard as an element of A. Since A is purely infinite and simple and $1 - p \neq 0$, there is a nonzero homomorphism $\varphi_1 : M_n \to (1-p)A(1-p)$. Define $\varphi : M_n \to A$ by $\varphi(b) = \varphi_0(b) + \varphi_1(b)$. Then we still have $T(b) = t^* \varphi(b) t$ for all $b \in M_n$.

Use Lemma 1.5 to choose a partial isometry $s_0 \in A$ such that $s_0^* s_0 = \varphi(1)$ and $\|s_0^* a s_0 - \varphi(S(a))\| < \varepsilon$ for all $a \in F$. Set $s = s_0 t$. Then for $a \in F$ we have

$$\|s^* a s - T(S(a))\| = \|t^* s_0^* a s_0 t - t^* \varphi(S(a)) t\| < \varepsilon.$$

Moreover, s is an isometry because $s_0^* s_0 \geq t_1 t_1^* \geq t t^*$, and it is not unitary because in fact $s_0^* s_0 > t_1 t_1^*$. ∎

The following lemma will be used to control the completely bounded norms of perturbations of maps on finite dimensional operator spaces. For now, we only need to know that the map W of the lemma satisfies $\|W\|_{\mathrm{cb}}$, $\|W^{-1}\|_{\mathrm{cb}} < 1 + \varepsilon$ for $\|a_l - b_l\|$ small enough, which is contained in the proof of Proposition 2.6 of [JP]. The more explicit estimate will be relevant in Section 6.

Lemma 1.8. Let A be a unital C*-algebra, let $a_1, \ldots, a_m \in A$ be linearly independent, and assume that $E = \mathrm{span}(a_1, \ldots, a_m)$ is unital and selfadjoint (and hence an operator system in A). Define

$$M = \sup \left\{ \max_{1 \leq l \leq m} |\alpha_l| : \left\| \sum_{l=1}^{m} \alpha_l a_l \right\| \leq 1 \right\}.$$

Then for $b_1, \ldots, b_m \in A$, the linear map $W : E \to \mathrm{span}(b_1, \ldots, b_m)$, given by $W(a_l) = b_l$, satisfies

$$\|W\|_{\mathrm{cb}} \leq 1 + mM \sum_{l=1}^{m} \|a_l - b_l\|$$

and, if $mM\sum_{l=1}^{m}\|a_l-b_l\|<1$, then

$$\|W^{-1}\|_{\mathrm{cb}} \leq \left(1 - mM\sum_{l=1}^{m}\|a_l-b_l\|\right)^{-1}.$$

Proof: Give $\mathbb{C}^m$ the norm $\|(\alpha_1,\ldots,\alpha_m)\|_\infty = \max_{1\leq l\leq m}|\alpha_l|$. Define $Q: E\to\mathbb{C}^m$ by sending a_l to the l-th standard basis vector ξ_l, and define $R: \mathbb{C}^m\to A$ by $R(\xi_l)=b_l-a_l$. Then $\|Q\|=M$ and $\|R\|\leq\sum_{l=1}^{m}\|a_l-b_l\|$. We have, using Lemma 2.3 of [EH],

$$\|R\circ Q\|_{\mathrm{cb}} \leq m\|R\circ Q\| \leq mM\sum_{l=1}^{m}\|a_l-b_l\|.$$

Since $W(a)=a+R(Q(a))$, the estimate on $\|W\|_{\mathrm{cb}}$ is now immediate. We further have, for any n and any $a\in M_n\otimes E$,

$$\|(\mathrm{id}_{M_n}\otimes W)(a)\| \geq \|a\| - \|[\mathrm{id}_{M_n}\otimes(R\circ Q)](a)\| \geq \|a\|(1-\|R\circ Q\|_{\mathrm{cb}}).$$

Therefore

$$\|(\mathrm{id}_{M_n}\otimes W)^{-1}\| \leq \left(1 - mM\sum_{l=1}^{m}\|a_l-b_l\|\right)^{-1}$$

for all n. ∎

We will need the following variant of an argument from one of the proofs of [Kr3].

Lemma 1.9. Let A be a unital C*-algebra, let $E\subset A$ be an operator system, let H be a Hilbert space, and let $S: E\to L(H)$ be a unital selfadjoint completely bounded map. Then there exists a unital completely positive map $T: A\to L(H)$ such that $\|T|_E - S\|_{\mathrm{cb}} \leq \|S\|_{\mathrm{cb}}-1$.

Proof: Wittstock's generalization of the Arveson extension theorem (see Theorem 7.2 of [Pl]) provides a linear map $Q: A\to L(H)$ such that $\|Q\|_{\mathrm{cb}}=\|S\|_{\mathrm{cb}}$ and $Q|_E=S$. Replacing Q by $x\mapsto\frac{1}{2}(Q(x)+Q(x^*)^*)$, we may assume in addition that Q is selfadjoint. Now apply Proposition 1.19 of [Ws] (see also the original version in the proof of Theorem 4.1 of [Kr3]) to obtain a unital completely positive map $T: A\to L(H)$ such that $\|T-Q\|_{\mathrm{cb}}\leq\|Q\|_{\mathrm{cb}}-1$. Then also $\|T|_E-S\|_{\mathrm{cb}}\leq\|S\|_{\mathrm{cb}}-1$. ∎

The following lemma is the second main technical result of this section.

Lemma 1.10. Let A be a separable unital exact C*-algebra, let E be a finite dimensional operator system in A, and let $\varepsilon>0$. For every $\delta<\varepsilon$ there exists an integer n such that whenever B_1 and B_2 are separable unital C*-algebras, with B_2 nuclear, and $V: E\to B_1$ and $W: E\to B_2$ are two unital completely positive maps such that V is injective and $V^{-1}: V(E)\to E$ satisfies $\|V^{-1}\otimes\mathrm{id}_{M_n}\|\leq 1+\delta$, then there is a unital completely positive map $T: B_1\to B_2$ such that $\|T\circ V-W\|<\varepsilon$.

Proof: Let $\rho=(\varepsilon-\delta)/[2(1+\delta)]>0$. Since A is exact, it has a nuclear embedding in $L(H)$ for some Hilbert space H. (See [Kr3], [Kr4], or Theorem 9.1 of [Ws].) We may thus assume that A is a unital subalgebra of $L(H)$ with the inclusion map nuclear. Let $\{a_1,\ldots,a_m\}$ be a basis for E. Choose $\mu>0$ small enough (using the previous lemma) that if $b_1,\ldots,b_m\in L(H)$ satisfy $\|a_l-b_l\|<\mu$, then the map $T(a_l)=b_l$, from E to $\mathrm{span}(b_1,\ldots,b_m)$, satisfies $\|T^{-1}\|_{\mathrm{cb}}<1+\rho$. By nuclearity of the inclusion, there are n and unital completely positive maps $S_1: E\to M_n$ and $\tilde{S}_2: M_n\to L(H)$ such that the elements $b_l=\tilde{S}_2\circ S_1(a_l)$ satisfy $\|a_l-b_l\|<\mu$. Let T be as above, using this choice of $b_1,\ldots,b_m$, and let $F=S_1(E)$, which is an operator space in M_n. Define $S_2: F\to E$ by $S_2=T^{-1}\circ\tilde{S}_2$. Then S_2 is unital, $S_2\circ S_1=\mathrm{id}_E$, and $\|S_2\|_{\mathrm{cb}}<1+\rho$. Moreover, from $S_1(x^*)=S_1(x)^*$ for $x\in E$, we get $S_2(y^*)=S_2(y)^*$ for $y\in F$.

Further choose, using the nuclearity of B_2, an integer r and unital completely positive maps $W_1: E\to M_r$ and $W_2: M_r\to B_2$ such that $\|W_2\circ W_1-W\|<\rho$. Since F is a subspace of M_n and $\|W_1\circ S_2\|_{\mathrm{cb}}<1+\rho$, Lemma 1.9 provides a unital completely positive map $Q: M_n\to M_r$ such that $\|Q|_F-W_1\circ S_2\|<\rho$.

Now consider $S_1\circ V^{-1}$, which is a linear map from $V(E)$ to M_n. Since S_1 is unital and completely positive, we have

$$\|(S_1\circ V^{-1})\otimes\mathrm{id}_{M_n}\| \leq \|S_1\|_{\mathrm{cb}}\|V^{-1}\otimes\mathrm{id}_{M_n}\| \leq 1+\delta.$$

By Proposition 7.9 of [Pl], this implies that $\|S_1 \circ V^{-1}\|_{\mathrm{cb}} \leq 1+\delta$. Also, $S_1 \circ V^{-1}$ is unital and selfadjoint. Applying Lemma 1.9 again, we obtain a unital completely positive map $R: B_1 \to M_n$ such that $\|R|_{V(E)} - S_1 \circ V^{-1}\| \leq \delta$. We then have

$$\begin{aligned} & \|W_2 \circ Q \circ R|_{V(E)} - W \circ V^{-1}\| \\ & \quad \leq \quad \|W - W_2 \circ W_1\| \|V^{-1}\| + \|W_2\| \|Q \circ R|_{V(E)} - W_1 \circ S_2 \circ S_1 \circ V^{-1}\| \\ & \quad \leq \quad \rho(1+\delta) + \|R|_{V(E)} - S_1 \circ V^{-1}\| + \|Q|_F - W_1 \circ S_2\| \|S_1 \circ V^{-1}\| \\ & \quad < \quad \rho(1+\delta) + \delta + \rho(1+\delta) = \varepsilon. \end{aligned}$$

Thus, $T = W_2 \circ Q \circ R$ is a unital completely positive map from B_1 to B_2 whose restriction to $V(E)$ differs in norm from $W \circ V^{-1}$ by less than ε. ∎

The following result is known in the important special case in which the right hand side of the inequality is zero. It is easy to give an example to show that the exponent $\frac{1}{2}$ can't be improved, but this also follows from the example after the next (more significant) lemma. It turns out that it is even impossible to improve the constant $\sqrt{2}$.

Lemma 1.11. Let A be a unital C*-algebra, let $u \in A$ be unitary, and let $s \in A$ be an isometry with range projection $e = ss^*$. Then

$$\|u - [eue + (1-e)u(1-e)]\| \leq \inf\{(2\|s^*us - v\|)^{1/2} : v \in A \text{ unitary}\}.$$

Proof: We prove that if $v \in A$ is any other unitary, then

$$\|u - [eue + (1-e)u(1-e)]\| \leq \sqrt{2\|s^*us - v\|}.$$

The element svs^* is a unitary in eAe and

$$\|eue - svs^*\| = \|s^*sus^*s - svs^*\| \leq \|s^*us - v\|.$$

So $\|(eue)^*(eue) - e\| \leq 2\|s^*us - v\|$. Now

$$e = eu^*ue = (eue)^*(eue) + [(1-e)ue]^*[(1-e)ue].$$

Therefore

$$\|[(1-e)ue]^*[(1-e)ue]\| \leq 2\|s^*us - v\|,$$

so that $\|(1-e)ue\| \leq \sqrt{2\|s^*us - v\|}$. Similarly, using $uu^* = 1$ instead of $u^*u = 1$, we get $\|eu(1-e)\| \leq \sqrt{2\|s^*us - v\|}$. Since e is orthogonal to $1-e$, it follows that

$$\|u - [eue + (1-e)u(1-e)]\| = \|(1-e)ue + eu(1-e)\| \leq \sqrt{2\|s^*us - v\|}.$$

∎

Lemma 1.12. Let A be a unital C*-algebra, let s and t be two isometries in A, and let D be a unital subalgebra of A which is isomorphic to $\mathcal{O}_2$ and such that every element of D commutes with s and t. Then there is a unitary $z \in A$ such that whenever u and v are unitaries in A commuting with every element of D, then

$$\|z^*uz - v\| \leq 11 \left[\max(\|s^*us - v\|, \|t^*vt - u\|) \right]^{1/2}.$$

Proof: Let B be the relative commutant of D in A. Then s and t, along with all possible choices of u and v, are in B. Since $\mathcal{O}_2$ is nuclear, there is a homomorphism from $\mathcal{O}_2 \otimes B$ to A which is the identity on B and sends $\mathcal{O}_2$ to D. Therefore we may as well take $A = \mathcal{O}_2 \otimes B$, with $s, t \in B$. We have to show that there is a unitary $z \in \mathcal{O}_2 \otimes B$ such that whenever u and v are unitaries in B, then

$$\|z(1 \otimes u)z^* - 1 \otimes v\| \leq 11 \left[\max(\|s^*us - v\|, \|t^*vt - u\|) \right]^{1/2}.$$

The unitary z will chop $1 \otimes u$ in pieces and reassemble them in a different way. To construct it, we start by defining an assortment of projections and partial isometries. Define

$$e_1 = ss^* \quad \text{and} \quad f_1 = tt^*,$$

and further define

$$e_2 = sf_1s^* \leq e_1, \quad f_2 = te_1t^* \leq f_1, \quad \text{and} \quad f_3 = te_2t^* \leq f_2.$$

Then define two sets of mutually orthogonal projections summing to 1 by

$$p_1 = 1 - e_1, \quad p_2 = e_1 - e_2, \quad \text{and} \quad p_3 = e_2$$

and

$$q_1 = 1 - f_1, \quad q_2 = f_1 - f_2, \quad q_3 = f_2 - f_3, \quad \text{and} \quad q_4 = f_3.$$

Next, construct partial isometries

$$c_1 = p_2sq_1, \quad c_2 = p_1t^*q_2, \quad c_3 = p_2t^*q_3, \quad \text{and} \quad c_4 = p_3t^*q_4.$$

One checks that

$$c_1^*c_1 = q_1 \quad \text{and} \quad c_1c_1^* = p_2,$$

while

$$c_j^*c_j = q_j \quad \text{and} \quad c_jc_j^* = p_{j-1}$$

for $j = 2, 3, 4$. Let s_1 and s_2 be the standard generators of $\mathcal{O}_2$. Define

$$z = s_1 \otimes c_1 + 1 \otimes c_2 + s_2 \otimes c_3 + 1 \otimes c_4,$$

which is easily checked to be a unitary in $\mathcal{O}_2 \otimes B$ such that

$$z(1 \otimes q_1)z^* = s_1s_1^* \otimes p_2, \;\; z(1 \otimes q_2)z^* = 1 \otimes p_1, \;\; z(1 \otimes q_3)z^* = s_2s_2^* \otimes p_2, \;\; \text{and} \;\; z(1 \otimes q_4)z^* = 1 \otimes p_3.$$

Now let $u, v \in B$ be unitaries. Set $\delta = \max(\|s^*us - v\|, \|t^*vt - u\|)$. Using $sq_1 = p_2s$ and $p_2 \leq ss^*$, we obtain

$$\begin{aligned} \|c_1(q_1vq_1)c_1^* - p_2up_2\| &= \|p_2svs^*p_2 - p_2up_2\| \\ &= \|p_2svs^*p_2 - p_2ss^*uss^*p_2\| \leq \|v - s^*us\| \leq \delta. \end{aligned}$$

Similarly, for $j = 2, 3, 4$ one gets

$$\|c_j(q_jvq_j)c_j^* - p_{j-1}up_{j-1}\| \leq \|t^*vt - u\| \leq \delta.$$

One checks that

$$\begin{aligned} &z[1 \otimes (q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4)]z^* \\ &\quad = \quad s_1s_1^* \otimes c_1(q_1vq_1)c_1^* + 1 \otimes c_2(q_2vq_2)c_2^* + s_2s_2^* \otimes c_3(q_3vq_3)c_3^* + 1 \otimes c_4(q_4vq_4)c_4^*, \end{aligned}$$

so that (using $s_1s_1^* + s_2s_2^* = 1$)

$$\|z(1 \otimes v)z^* - 1 \otimes u\| \leq \delta + \|q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\| + \|p_1up_1 + p_2up_2 + p_3up_3 - u\|.$$

We now have to estimate the last two terms in the last inequality above. Recall that $e_1 = ss^*$, so that

$$\|u - [e_1ue_1 + (1 - e_1)u(1 - e_1)]\| \leq \sqrt{2\|s^*us - v\|} \leq \sqrt{2\delta}$$

by Lemma 1.11. Since $e_2 = stt^*s^*$, we get

$$\|e_2ue_2 - stut^*s^*\| \leq \|t^*s^*ust - u\| \leq \|s^*us - v\| + \|t^*vt - u\| \leq 2\delta.$$

So

$$\|u - [e_2ue_2 + (1 - e_2)u(1 - e_2)]\| \leq \sqrt{4\delta},$$

again by Lemma 1.11. Compressing by $e_1 \geq e_2$, we get

$$\|e_1ue_1 - [e_2ue_2 + (e_1 - e_2)u(e_1 - e_2)]\| \leq \sqrt{4\delta}.$$

Recalling the definitions of p_1, p_2, and p_3, it follows that

$$\|p_1up_1 + p_2up_2 + p_3up_3 - u\| \leq \left(\sqrt{2} + \sqrt{4}\right)\sqrt{\delta}.$$

A similar sequence of estimates, with one more step and using the equations $f_1 = tt^*$, $f_2 = tss^*t^*$, and $f_3 = (tst)(tst)^*$, gives

$$\|q_1vq_1 + q_2vq_2 + q_3vq_3 + q_4vq_4 - v\| \leq \left(\sqrt{2} + \sqrt{4} + \sqrt{6}\right)\sqrt{\delta}.$$

We note that necessarily $\delta \leq 2$, so that $\delta \leq \sqrt{2\delta}$. We can therefore put our estimates together to get

$$\|z^*(1 \otimes u)z - 1 \otimes v\| = \|z(1 \otimes v)z^* - 1 \otimes u\| \leq \left[\sqrt{2} + \left(\sqrt{2} + \sqrt{4}\right) + \left(\sqrt{2} + \sqrt{4} + \sqrt{6}\right)\right] \sqrt{\delta} \leq 11\sqrt{\delta}.$$

∎

The exponent $\frac{1}{2}$ in this lemma can't be improved, as we now show by example, even if z is allowed to depend on u and v. It follows from the proof of the lemma that the exponent $\frac{1}{2}$ can't be improved in the previous lemma either. (We will also see a more indirect proof of this below: any improvement here would imply a a corresponding improvement in the exponent in Proposition 4.13, but Remark 6.11 shows that no improvement is possible there.)

Example 1.13. Let $D = M_2 \otimes \mathcal{O}_2$. Let s_1 and s_2 be the two standard generating isometries of $\mathcal{O}_2$, and set $p_j = s_j s_j^*$. Let $\alpha \in \mathbb{R}$, and define a unitary $w_\alpha \in M_2$ by

$$w_\alpha = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) \\ -\sin(\alpha) & \cos(\alpha) \end{pmatrix}.$$

Use the fact that any two nonzero projections in $M_2 \otimes \mathcal{O}_2$ are Murray-von Neumann equivalent to choose $c \in M_2 \otimes \mathcal{O}_2$ such that

$$c^*c = 1 \otimes p_2 \quad \text{and} \quad cc^* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes p_2.$$

Then set

$$u = w_\alpha \otimes 1, \quad v = w_\alpha \otimes p_1 + 1 \otimes p_2, \quad s = 1 \otimes p_1 + c, \quad \text{and} \quad t = 1 \otimes s_1.$$

Then

$$s^*us = w_\alpha \otimes p_1 + \cos(\alpha)(1 \otimes p_2) \quad \text{and} \quad t^*vt = u.$$

Therefore

$$\max(\|s^*us - v\|, \|t^*vt - u\|) = 1 - \cos(\alpha).$$

Also

$$\mathrm{sp}(u) = \{\exp(i\alpha), \exp(-i\alpha)\} \quad \text{and} \quad \mathrm{sp}(v) = \{\exp(i\alpha), 1, \exp(-i\alpha)\}.$$

Therefore, if A is any C*-algebra at all which contains D as a unital subalgebra, and if z is any unitary in A, we have

$$\|z^*uz - v\| \geq |\exp(i\alpha) - 1| = \sqrt{2}(1 - \cos(\alpha))^{1/2}.$$

∎

Our first application of the technical results of this section is the following lemma and theorem.

Lemma 1.14. Let A be a separable unital exact C*-algebra, and let B be a separable nuclear unital purely infinite simple C*-algebra. Let $\varphi, \psi : A \to B$ be two injective unital homomorphisms. Then the homomorphisms from A to $\mathcal{O}_2 \otimes B$, given by $a \mapsto 1 \otimes \varphi(a)$ and $a \mapsto 1 \otimes \psi(a)$, are approximately unitarily equivalent.

Proof: Let $u_1, \ldots, u_n \in A$ be unitaries, and let $\varepsilon > 0$. We prove that there is a unitary $z \in \mathcal{O}_2 \otimes B$ such that

$$\|z(1 \otimes \varphi(u_j))z^* - 1 \otimes \psi(u_j)\| < \varepsilon$$

for $1 \leq j \leq n$. Let

$$E = \mathrm{span}\{1, u_1, u_1^*, \ldots, u_n, u_n^*\},$$

which is a finite dimensional operator system. Lemma 1.10 applies to $\varphi|_E$ and $\psi|_E$ (with $\delta = 0$), and provides unital completely positive maps $S, T : B \to B$ such that

$$\|S \circ \varphi|_E - \psi|_E\| < \frac{1}{2}\left(\frac{\varepsilon}{11}\right)^2 \quad \text{and} \quad \|T \circ \psi|_E - \varphi|_E\| < \frac{1}{2}\left(\frac{\varepsilon}{11}\right)^2.$$

In particular,

$$\|S(\varphi(u_j)) - \psi(u_j)\| < \frac{1}{2}\left(\frac{\varepsilon}{11}\right)^2 \quad \text{and} \quad \|T(\psi(u_j)) - \varphi(u_j)\| < \frac{1}{2}\left(\frac{\varepsilon}{11}\right)^2$$

for $1 \leq j \leq n$. Proposition 1.7 then gives isometries s, $t \in B$ such that

$$\|s^* \varphi(u_j) s - \psi(u_j)\| < \left(\frac{\varepsilon}{11}\right)^2 \quad \text{and} \quad \|t^* \psi(u_j) t - \varphi(u_j)\| < \left(\frac{\varepsilon}{11}\right)^2$$

for $1 \leq j \leq n$. The existence of the required unitary $z \in \mathcal{O}_2 \otimes B$ now follows from Lemma 1.12. ∎

As a corollary, we obtain:

Theorem 1.15. Let A be a separable unital exact C*-algebra. Then any two injective unital homomorphisms from A to $\mathcal{O}_2$ are approximately unitarily equivalent (Definition 0.5).

Proof: Let $\varphi, \psi : A \to \mathcal{O}_2$ be injective and unital. Let $\mu : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism (from Theorem 0.8), and let $\beta : \mathcal{O}_2 \to \mathcal{O}_2 \otimes \mathcal{O}_2$ be $\beta(a) = 1 \otimes a$. Then $\mu \circ \beta$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$ by Proposition 0.7. Also, $\beta \circ \varphi$ is approximately unitarily equivalent to $\beta \circ \psi$ by Lemma 1.14. Thus φ is approximately unitarily equivalent to $\mu \circ \beta \circ \varphi$, which is approximately unitarily equivalent to $\mu \circ \beta \circ \psi$, which in turn is approximately unitarily equivalent to ψ. ∎

2. Embedding in $\mathcal{O}_2$

The main result of this section is that every separable unital exact C*-algebra can be unitally embedded in $\mathcal{O}_2$. The algebra of bounded sequences modulo sequences converging to zero plays an important role in the proof, so we begin by establishing notation concerning it.

Notation 2.1. For any C*-algebra D, we denote by $l^\infty(D)$ the set of bounded sequences $d = (d_1, d_2, \dots)$ with values in D. It is a C*-algebra with the obvious operations and norm. For compatibility with the notation for ultrapowers below, we define $c_\infty(D) = C_0(\mathbb{N}) \otimes D \subset l^\infty(D)$ and $D_\infty = l^\infty(D)/c_\infty(D)$. We denote by $\pi_\infty^{(D)}$ (or π_∞ when D is clear from the context) the quotient map $l^\infty(D) \to D_\infty$.

The technical results of the previous section imply fairly directly that if A is separable and exact, and has an embedding in $(\mathcal{O}_2)_\infty$ which lifts to a unital completely positive map to $l^\infty(\mathcal{O}_2)$, then A has an embedding in $\mathcal{O}_2$. In particular, this applies to quasidiagonal exact C*-algebras. The rest of the section is devoted to the extension from the quasidiagonal case to the general case. It is possible to embed any separable C*-algebra in a crossed product of a quasidiagonal separable C*-algebra by $\mathbb{Z}$, and the method is to show that crossed products by $\mathbb{Z}$ of exact C*-algebras embeddable in $\mathcal{O}_2$ are again embeddable in $\mathcal{O}_2$.

Lemma 2.2. Let A be a unital separable exact C*-algebra. If there is a unital injective homomorphism from A to $(\mathcal{O}_2)_\infty = l^\infty(\mathcal{O}_2)/c_\infty(\mathcal{O}_2)$ which has a lifting to a unital completely positive map from A to $l^\infty(\mathcal{O}_2)$, then there is an injective unital homomorphism from A to $\mathcal{O}_2$.

Proof: Let $\varphi : A \to (\mathcal{O}_2)_\infty$ be a unital injective homomorphism which has a lifting as in the hypotheses of the lemma. Let $u_1, u_2, \dots$ be a sequence of unitaries in A whose linear span is dense in A. Define finite dimensional operator systems E_n by $E_n = \mathrm{span}\{1, u_1, u_1^*, \dots, u_n, u_n^*\}$. Then $\mathbb{C} \cdot 1 = E_0 \subset E_1 \subset \cdots \subset A$ and $\overline{\bigcup_{n=0}^\infty E_n} = A$. We first show that there is a unital injective homomorphism $\psi : A \to (\mathcal{O}_2)_\infty$ with a unital completely positive lifting $a \mapsto V(a) = (V_1(a), V_2(a), \dots)$ from A to $l^\infty(\mathcal{O}_2)$ with the following property: For each fixed n, for all sufficiently large m (how large depends on n), the restriction $V_m|_{E_n}$ is injective, and furthermore its inverse, defined on $V_m(E_n)$, satisfies $\lim_{m\to\infty} \|(V_m|_{E_n})^{-1} \otimes \mathrm{id}_{M_k}\| = 1$ for all $k \in \mathbb{N}$.

To do this, let $a \mapsto Q(a) = (Q_1(a), Q_2(a), \dots)$ be a lifting of φ to a unital completely positive map from A to $l^\infty(\mathcal{O}_2)$. Injectivity of φ does not imply that $\lim_{m\to\infty} \|(Q_m \otimes \mathrm{id}_{M_k})(a)\| = \|a\|$, but we can remedy this by grouping the Q_m together in blocks. Injectivity of φ does imply that for every $N \in \mathbb{N}$, the map $a \mapsto \varphi^{(N)}(a) = \pi_\infty(Q_{N+1}(a), Q_{N+2}(a), \dots)$ is again an injective homomorphism. Therefore, for every N, $k \in \mathbb{N}$ and $a \in M_k \otimes A$, we have

$$\lim_{m\to\infty} \|((Q_{N+1} \otimes \mathrm{id}_{M_k})(a), \dots, (Q_{N+m} \otimes \mathrm{id}_{M_k})(a))\| = \|(\varphi^{(N)} \otimes \mathrm{id}_{M_k})(a)\| = \|a\|.$$

Since each E_n is finite dimensional, we can therefore construct by induction a sequence

$$0 = N_1 < N_2 < \cdots < N_m < N_{m+1} < \cdots$$

of integers such that

$$\|((Q_{N_m+1} \otimes \mathrm{id}_{M_k})(a), \ldots, (Q_{N_{m+1}} \otimes \mathrm{id}_{M_k})(a))\| \geq (1 - 2^{-m})\|a\|$$

for $k \leq m$ and $a \in M_k \otimes E_m$. Let $\sigma_m : \mathcal{O}_2^{N_{m+1}-N_m} \to \mathcal{O}_2$ be any unital homomorphism. Define $V_m : A \to \mathcal{O}_2$ by

$$V_m(a) = \sigma_m((Q_{N_m+1}(a), \ldots, Q_{N_{m+1}}(a)).$$

Note that V_m is unital and completely positive because each Q_j is, so that also $V(a) = (V_1(a), V_2(a), \ldots)$ defines a unital completely positive map from A to $l^\infty(\mathcal{O}_2)$. By construction we have $\lim_{m\to\infty} \|(V_m|_{E_n})^{-1} \otimes \mathrm{id}_{M_k}\| = 1$ for each fixed k, $n \in \mathbb{N}$. Taking $k = 1$, we see that the linear map $\psi = \pi_\infty \circ V$ is isometric, hence injective. Since $\lim_{j\to\infty}(Q_j(ab) - Q_j(a)Q_j(b)) = 0$ for a, $b \in A$, we also obtain $\lim_{m\to\infty}(V_m(ab) - V_m(a)V_m(b)) = 0$ for a, $b \in A$. Therefore ψ is a homomorphism, and its lifting V satisfies the required conditions.

Choose numbers $\delta_m > 0$ such that $\delta_0 \geq \delta_1 \geq \cdots$ and $2\delta_m + 11\sqrt{5\delta_m} < 2^{-m}$. Using Lemma 1.10, choose positive integers $k(m)$ with $k(0) \leq k(1) \leq \cdots$ and such that whenever V, $W : E_m \to \mathcal{O}_2$ are unital completely positive with V injective and $\|V^{-1} \otimes \mathrm{id}_{M_{k(m)}}\| \leq 1 + \delta_m$, then there is a unital completely positive map $T : \mathcal{O}_2 \to \mathcal{O}_2$ such that $\|T \circ V - W\| < 2\delta_m$. The conditions on V imply that we can pass to a subsequence in the variable m in such a way that $V_m|_{E_n}$ is injective for $n \leq m$, and moreover we have the estimates

$$\|(V_m|_{E_n})^{-1} \otimes \mathrm{id}_{M_{k(m)}}\| \leq 1 + \delta_m, \quad \|V_m(u_n)^* V_m(u_n) - 1\| < \delta_m, \quad \text{and} \quad \|V_m(u_n)V_m(u_n)^* - 1\| < \delta_m$$

for all m and all $n \leq m$.

Using these estimates and Lemma 1.10, find unital completely positive maps S_m, $T_m : \mathcal{O}_2 \to \mathcal{O}_2$ such that

$$\|T_m \circ V_m|_{E_m} - V_{m+1}|_{E_m}\| \leq 2\delta_m \quad \text{and} \quad \|S_m \circ V_{m+1}|_{E_m} - V_m|_{E_m}\| \leq 2\delta_m.$$

For $1 \leq j \leq m$ define unitaries $x_m^{(j)} = V_m(u_j)[V_m(u_j)^* V_m(u_j)]^{-1/2}$. Then $\|x_m^{(j)} - V_m(u_j)\| \leq \delta_m$ by Lemma 0.1. It follows that

$$\|T_m(x_m^{(j)}) - x_{m+1}^{(j)}\| \leq 4\delta_m \quad \text{and} \quad \|S_m(x_{m+1}^{(j)}) - x_m^{(j)}\| \leq 4\delta_m.$$

Proposition 1.7 gives isometries s_m, $t_m \in \mathcal{O}_2$ (depending on m) such that

$$\|s_m^* x_m^{(j)} s_m - x_{m+1}^{(j)}\| \leq 5\delta_m \quad \text{and} \quad \|t_m^* x_{m+1}^{(j)} t_m - x_m^{(j)}\| \leq 5\delta_m$$

for $1 \leq j \leq m$. Lemma 1.12 now gives unitaries $z_m \in \mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$\|z_m(1 \otimes x_m^{(j)})z_m^* - 1 \otimes x_{m+1}^{(j)}\| \leq 11\sqrt{5\delta_m}.$$

Thus

$$\|z_m(1 \otimes V_m(u_j))z_m^* - 1 \otimes V_{m+1}(u_j)\| \leq 2\delta_m + 11\sqrt{5\delta_m} < 2^{-m}$$

for $1 \leq j \leq m$.

Now define $y_n = z_1^* z_2^* \cdots z_n^*$. Then the y_n are unitaries such that $\lim_{n\to\infty} y_n(1 \otimes V_n(u_j))y_n^*$ exists for all j. It follows that the limit $\psi_0(a) = \lim_{n\to\infty} y_n(1 \otimes V_n(a))y_n^*$ exists for all $a \in \bigcup_{n=1}^\infty E_n$. Furthermore, for each n and m, the map $V_m|_{E_n}$ is unital and completely positive. Therefore $\psi_0|_{E_n}$ is unital and completely positive, whence $\|\psi_0|_{E_n}\| \leq 1$. It follows that ψ_0 extends by continuity to a unital completely positive map $\psi : A \to \mathcal{O}_2 \otimes \mathcal{O}_2$. Since $\lim_{m\to\infty}(V_m(ab) - V_m(a)V_m(b)) = 0$ for all $a \in \bigcup_{n=1}^\infty E_n$, it follows that ψ is actually a homomorphism. Finally, for $a \in \bigcup_{n=1}^\infty E_n$ we have $\|\psi(a)\| = \lim_{n\to\infty} \|V_m(a)\| = \|a\|$, from which it follows that ψ is isometric and hence injective.

We thus have an injective unital homomorphism from A to $\mathcal{O}_2 \otimes \mathcal{O}_2$. The existence of an injective unital homomorphism from A to $\mathcal{O}_2$ now follows from the isomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ (Theorem 0.8). ∎

Recall that a separable C*-algebra A is called *quasidiagonal* (weakly quasidiagonal in some papers) if there is an injective representation π of A on a separable Hilbert space H and a sequence $p_1 \leq p_2 \leq \cdots$ of finite rank projections on H such that $p_n \to 1$ in the strong operator topology and $\lim_{n\to\infty} \|p_n\pi(a) - \pi(a)p_n\| = 0$ for all $a \in A$.

Corollary 2.3. Let A be a separable unital exact quasidiagonal C*-algebra. Then there exists a unital injective homomorphism from A to $\mathcal{O}_2$.

Proof: By the previous lemma, it suffices to find a unital injective homomorphism from A to $l^\infty(\mathcal{O}_2)/c_0(\mathcal{O}_2)$ which has a lifting to a unital completely positive map from A to $l^\infty(\mathcal{O}_2)$. Now $\mathcal{O}_2$ contains a unital copy of every matrix algebra M_k, so one can easily construct a unital injective homomorphism from any product $\prod_{n=1}^\infty M_{k(n)}$ to $l^\infty(\mathcal{O}_2)$. (The product $\prod_{n=1}^\infty M_{k(n)}$ means, of course, the C*-algebra of all bounded sequences in the set-theoretic product.) Therefore there is a unital injective homomorphism from $\prod_{n=1}^\infty M_{k(n)}/\bigoplus_{n=1}^\infty M_{k(n)}$ to $l^\infty(\mathcal{O}_2)/c_0(\mathcal{O}_2)$. So it suffices to find a unital injective homomorphism from A to $\prod_{n=1}^\infty M_{k(n)}/\bigoplus_{n=1}^\infty M_{k(n)}$, with a lifting to a unital completely positive map from A to $\prod_{n=1}^\infty M_{k(n)}$, for some sequence $k(1), k(2), \dots$ of positive integers. This is easy to do, and is done in Proposition 3.1.3 and the preceding remark in [BK]. ∎

We now start our preparations for the general case. The following lemma is a variant of a result of Effros and Haagerup [EH], and will be used to construct the lifting in the hypotheses of Lemma 2.2.

Lemma 2.4. Let A and B be separable unital C*-algebras, let J be an ideal in A which is approximately injective in the sense of [EH] (definition before Lemma 3.3), and let $\varphi : A \to B/J$ be an injective homomorphism. Let H be a separable infinite dimensional Hilbert space, and suppose that the induced map of algebraic tensor products $A \otimes_{\mathrm{alg}} L(H) \to (B \otimes_{\mathrm{alg}} L(H))/(J \otimes_{\mathrm{alg}} L(H))$ extends continuously to a (necessarily injective) homomorphism

$$\overline{\varphi} : A \otimes_{\min} L(H) \to (B \otimes_{\min} L(H))/(J \otimes_{\min} L(H)).$$

Then there is a unital completely positive map $T : A \to B$ which lifts φ.

Proof: Let $\rho : B \to B/J$ be the quotient map.

We first reduce to the case $A = B/J$ and $\varphi = \mathrm{id}_{B/J}$. Let $B_0 = \rho^{-1}(\varphi(A)) \subset B$, and let $\rho_0 = \rho|_{B_0}$. Since the minimal tensor product preserves inclusions, we have $J \otimes_{\min} L(H) \subset B_0 \otimes_{\min} L(H) \subset B \otimes_{\min} L(H)$. So the hypotheses of the lemma hold with B_0 in place of B.

We now assume $A = B/J$. The desired conclusion will follow from Theorem 3.4 of [EH], provided we verify the hypothesis (b) there, that is, that for every unital C*-algebra C, the kernel of $\rho \otimes \mathrm{id}_C : B \otimes_{\min} C \to (B/J) \otimes_{\min} C$ is exactly $J \otimes_{\min} C$. The hypothesis of the lemma is that this is true for $C = L(H)$.

If C is separable, we may suppose $C \subset L(H)$, and use Proposition 2.6 of [Ws]. (Alternatively, combine the method below for reduction to the separable case with Lemma 3.9 of [Kr3].) For general C, we need only show that $\ker(\rho \otimes \mathrm{id}_C) \subset J \otimes_{\min} C$. Let $a \in \ker(\rho \otimes \mathrm{id}_C)$. Choose a separable subalgebra $C_0 \subset C$ such that $a \in B \otimes_{\min} C_0 \subset B \otimes_{\min} C$. Using $(B/J) \otimes_{\min} C_0 \subset (B/J) \otimes_{\min} C$, we have $\ker(\rho \otimes \mathrm{id}_C) = \ker(\rho \otimes \mathrm{id}_{C_0}) \cap B \otimes_{\min} C_0$, whence $a \in \ker(\rho \otimes \mathrm{id}_{C_0})$. So $a \in J \otimes_{\min} C_0 \subset J \otimes_{\min} C$ by the separable case. ∎

We now turn to the crossed product part of the construction.

Let $\alpha : G \to \mathrm{Aut}(A)$ be an action of a discrete group G on a C*-algebra A. By a *covariant representation* (u, φ) of the system (G, A, α) in a unital C*-algebra B, we mean the obvious generalization of a covariant representation on a Hilbert space. That is, $u : G \to U(B)$ is a homomorphism from G to the unitary group $U(B)$ of B, $\varphi : A \to B$ is a homomorphism of C*-algebras, and $u(g)\varphi(a)u(g)^* = \varphi(\alpha_g(a))$ for all $a \in A$ and $g \in G$.

The following lemma is the easy generalization to this context of standard results on regular representations of crossed products by discrete amenable groups.

Lemma 2.5. Let G be a discrete amenable group, and let $\alpha : G \to \mathrm{Aut}(A)$ be an action of G on a unital C*-algebra A. Let (u, φ) be a covariant representation of (G, A, α) in a unital C*-algebra B, with φ injective. For $g \in G$, let g also denote the corresponding elements of the group C*-algebra $C^*(G)$ and the crossed product C*-algebra $C^*(G, A, \alpha)$. Then there is an injective homomorphism $\psi : C^*(G, A, \alpha) \to C^*(G) \otimes B$ determined by $\psi(a) = 1 \otimes \varphi(a)$ for $a \in A$ and $\psi(g) = g \otimes u(g)$ for $g \in G$.

Proof: The existence (and uniqueness) of ψ is immediate from the universal property of the crossed product. We have to prove injectivity.

Let $\pi_0 : B \to L(H_0)$ be an injective representation of B on a Hilbert space H_0, and let λ be the regular representation of $C^*(G)$ on the Hilbert space $l^2(G)$. Then $\sigma = (\lambda \otimes \pi_0) \circ \psi$ is a representation of $C^*(G, A, \alpha)$ on the Hilbert space $H = l^2(G) \otimes H_0$. We will prove it is unitarily equivalent to the regular representation π of $C^*(G, A, \alpha)$ on H associated with the injective representation $\pi_0 \circ \varphi$ of A. (See Section 7.7 of [Pd].) This will prove injectivity of ψ, since G is amenable (so that π is injective by Theorem 7.7.5 of [Pd]).

The two representations are given by the formulas

$$(\pi(a)\xi)(g) = (\pi_0 \circ \varphi \circ \alpha_g^{-1})(a)(\xi(g)) \quad \text{and} \quad (\sigma(a)\xi)(g) = (\pi_0 \circ \varphi)(a)(\xi(g))$$

for $a \in A$, $\xi \in H$ (viewed as $l^2(G, H_0)$), and $g \in G$, and

$$\pi(g) = \lambda(g) \otimes 1 \quad \text{and} \quad \sigma(g) = \lambda(g) \otimes \pi_0(u(g))$$

for $g \in G$. Let v be the unitary on $l^2(G, H_0)$ given by $(v\xi)(g) = \pi_0(u(g))(\xi(g))$ for $\xi \in l^2(G, H_0)$ and $g \in G$. Then one can check directly that

$$v\pi(a)v^* = \sigma(a) \quad \text{and} \quad v\pi(g)v^* = \sigma(g)$$

for $a \in A$ and $g \in G$. So σ is unitarily equivalent to π, as desired. ∎

In applications of the following lemma, the approximate innerness assumption will be derived from Lemma 1.14.

Lemma 2.6. Let B a unital C*-algebra, let A be a subalgebra of B which contains the identity, and let $\sigma \in \mathrm{Aut}(A)$. Suppose that σ is approximately inner in B, that is, there is a sequence $v_1, v_2, \ldots$ of unitaries in B such that $\lim_{n\to\infty} v_n a v_n^* = \sigma(a)$ for all $a \in A$. Let z be the standard generator of $C(S^1)$ and let u be the canonical unitary in $C^*(\mathbb{Z}, A, \sigma)$ which implements σ on A. Then the maps

$$a \mapsto 1 \otimes \pi^B(a, a, \ldots) \quad \text{and} \quad u \mapsto z \otimes \pi^B(v_1, v_2, \ldots)$$

define an injective homomorphism $\varphi : C^*(\mathbb{Z}, A, \sigma) \to C(S^1) \otimes [l^\infty(B)/c_\infty(B)]$. Moreover, for any unital C*-algebra C, this homomorphism extends continuously to an injective homomorphism

$$C^*(\mathbb{Z}, A, \sigma) \otimes_{\min} C \to C(S^1) \otimes [(l^\infty(B) \otimes_{\min} C)/(c_\infty(B) \otimes_{\min} C)].$$

Proof: We first show that the last sentence in the lemma follows from the rest. Representing everything on Hilbert spaces and forming the spatial tensor and crossed products, we easily see that $C^*(\mathbb{Z}, A, \sigma) \otimes_{\min} C = C^*(\mathbb{Z}, A \otimes_{\min} C, \sigma \otimes \mathrm{id}_C)$. Moreover, clearly

$$\lim_{n\to\infty} (v_n \otimes 1)x(v_n \otimes 1)^* = (\sigma \otimes \mathrm{id}_C)(x)$$

for all $x \in A \otimes_{\min} C$. (Check on the algebraic tensor product.) The first part of the lemma (applied to both A and $A \otimes_{\min} C$) therefore implies that φ extends continuously to an injective homomorphism

$$\overline{\varphi} : C^*(\mathbb{Z}, A, \sigma) \otimes_{\min} C \to C(S^1) \otimes [l^\infty(B \otimes_{\min} C)/c_\infty(B \otimes_{\min} C)].$$

Now $l^\infty(B) \otimes_{\min} C$ is a subalgebra of $l^\infty(B \otimes_{\min} C)$. (Represent B faithfully on a Hilbert space H_1, and represent $l^\infty(B)$ faithfully on $l^2(\mathbb{N}) \otimes H_1$ in the obvious way. Represent C faithfully on another Hilbert space H_2, and compare the spatial tensor products as represented on $l^2(\mathbb{N}) \otimes H_1 \otimes H_2$.) Since

$$c_\infty(B) \otimes_{\min} C = c_\infty(\mathbb{C}) \otimes_{\min} B \otimes_{\min} C = c_\infty(B \otimes_{\min} C),$$

the inclusion of $l^\infty(B) \otimes_{\min} C$ in $l^\infty(B \otimes_{\min} C)$ gives an injective homomorphism

$$[l^\infty(B) \otimes_{\min} C]/[c_\infty(B) \otimes_{\min} C] \to l^\infty(B \otimes_{\min} C)/c_\infty(B \otimes_{\min} C).$$

One immediately checks that the range of $\overline{\varphi}$ is contained in the image of

$$C(S^1) \otimes \Big[[l^\infty(B) \otimes_{\min} C]/[c_\infty(B) \otimes_{\min} C] \Big].$$

This gives the desired extension.

We now prove the first part of the lemma. The hypotheses immediately imply that

$$(v_1, v_2, \ldots) \cdot (a, a, \ldots) \cdot (v_1, v_2, \ldots)^* - (\sigma(a), \sigma(a), \ldots) \in c_\infty(B)$$

for all $a \in A$. Therefore

$$a \mapsto \pi^B(a, a, \ldots) \quad \text{and} \quad u \mapsto \pi^B(v_1, v_2, \ldots)$$

define a homomorphism from $C^*(\mathbb{Z}, A, \sigma)$ to $l^\infty(B)/c_\infty(B)$. Moreover, $a \mapsto \pi^B(a, a, \ldots)$ is injective. So

$$a \mapsto 1 \otimes \pi^B(a, a, \ldots) \quad \text{and} \quad u \mapsto z \otimes \pi^B(v_1, v_2, \ldots)$$

define an injective homomorphism from $C^*(\mathbb{Z}, A, \sigma)$ to $C(S^1) \otimes [l^\infty(B)/c_\infty(B)]$ by Lemma 2.5. ∎

Lemma 2.7. Let B a separable nuclear unital C*-algebra, let A be a subalgebra of B which contains the identity, and let $\sigma \in \mathrm{Aut}(A)$ be approximately inner in B. Then the homomorphism $C^*(\mathbb{Z}, A, \sigma) \to C(S^1) \otimes [l^\infty(B)/c_\infty(B)]$ of the previous lemma has a lifting to a unital completely positive map $C^*(\mathbb{Z}, A, \sigma) \to C(S^1) \otimes l^\infty(B)$.

Proof: This now follows immediately from Lemma 2.4, using $C^*(\mathbb{Z}, A, \sigma)$ in place of A, $C(S^1) \otimes l^\infty(B)$ in place of B, and $C(S^1) \otimes c_\infty(B)$ in place of J. The ideal $C(S^1) \otimes c_\infty(B)$ is approximately injective because it is nuclear, and the extension to a homomorphism

$$C^*(\mathbb{Z}, A, \sigma) \otimes_{\min} L(H) \to [C(S^1) \otimes l^\infty(B) \otimes_{\min} L(H)]/[C(S^1) \otimes c_\infty(B) \otimes_{\min} L(H)]$$

is obtained from the previous lemma with $C = L(H)$. ∎

Theorem 2.8. Let A be a separable unital exact C*-algebra. Then there exists an injective unital homomorphism from A to $\mathcal{O}_2$.

Proof: The cone $C_0([0,1)) \otimes A$ is quasidiagonal by Theorem 5 of [Vc]. The C*-algebra $B_0 = (C_0(\mathbb{R}) \otimes A)^\dagger$ is the unitization of a subalgebra of $C_0([0,1)) \otimes A$, hence also quasidiagonal. It is still exact by Proposition 7.1 (iii) and (vi) of [Kr4]. (Also compare with Remark 4.4 (4) of [Ws].) Therefore Corollary 2.3 provides a unital embedding $\varphi_0 : B_0 \to \mathcal{O}_2$. Let $B = C^*(\mathbb{Z}, B_0, \tau)$, where the action τ is by translation on $\mathbb{R}$ and is trivial on A. We produce an embedding of B in $(\mathcal{O}_2)_\infty = l^\infty(\mathcal{O}_2)/c_\infty(\mathcal{O}_2)$ which has a lifting to a unital completely positive map from B to $l^\infty(\mathcal{O}_2)$.

Let τ_1 be the automorphism of B_0 which generates the action, and let $\psi_0 = \varphi_0 \circ \tau_1 : B_0 \to \mathcal{O}_2$. Let $\mu : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism, obtained from Theorem 0.8. Define $\varphi, \psi : B_0 \to \mathcal{O}_2$ by $\varphi(a) = \mu(\varphi_0(a) \otimes 1)$ and $\psi(a) = \mu(\psi_0(a) \otimes 1)$. Lemma 1.14 implies that ψ is approximately unitarily equivalent to φ. Thus, using the embedding φ of B_0 in $\mathcal{O}_2$, the automorphism τ_1 is approximately inner in $\mathcal{O}_2$ in the sense of Lemma 2.6. That lemma therefore provides an injective homomorphism from B to $C(S^1) \otimes (\mathcal{O}_2)_\infty$, which has a lifting to a unital completely positive map from B to $C(S^1) \otimes l^\infty(\mathcal{O}_2)$ by Lemma 2.7. It is easy to find an injective homomorphism from $C(S^1)$ to the 2^∞ UHF algebra D, and it follows from Corollary 7.5 of [Rr1] that $D \otimes \mathcal{O}_2 \cong \mathcal{O}_2$. We thus obtain an injective composite homomorphism

$$B \longrightarrow C(S^1) \otimes (\mathcal{O}_2)_\infty \longrightarrow D \otimes (\mathcal{O}_2)_\infty \longrightarrow (D \otimes \mathcal{O}_2)_\infty \xrightarrow{\cong} (\mathcal{O}_2)_\infty,$$

with a unital completely positive lifting given by

$$B \longrightarrow C(S^1) \otimes l^\infty(\mathcal{O}_2) \longrightarrow D \otimes l^\infty(\mathcal{O}_2) \longrightarrow l^\infty(D \otimes \mathcal{O}_2) \longrightarrow l^\infty(\mathcal{O}_2).$$

The crossed product $B = C^*(\mathbb{Z}, B_0, \tau)$ is still exact, by Proposition 7.1 (v) of [Kr4]. Lemma 2.2 therefore provides an injective unital homomorphism $\gamma : B \to \mathcal{O}_2$. Now B contains as a subalgebra $C^*(\mathbb{Z}, C_0(\mathbb{R}) \otimes A)) \cong C(S^1) \otimes K \otimes A$, and this subalgebra in turn contains an isomorphic copy A_0 of A. Let $p \in B$ be the identity of A_0. Then

$$\gamma|_{A_0} : A_0 \to \gamma(p)\mathcal{O}_2\gamma(p)$$

is a unital embedding of A in $\gamma(p)\mathcal{O}_2\gamma(p) \cong \mathcal{O}_2$, as desired. ∎

3. Tensor products with $\mathcal{O}_2$ and $\mathcal{O}_\infty$

In this section, we prove that if A is a simple separable unital nuclear C*-algebra, then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$, and that if, in addition, A is purely infinite, then $\mathcal{O}_\infty \otimes A \cong A$. The key technical point is that if A is separable, nuclear, unital, purely infinite, and simple, and if $\omega \in \beta\mathbb{N} - \mathbb{N}$, then the relative commutant of the image of A in the ultrapower A_ω, the algebra of bounded sequences in A modulo those that vanish at ω, is again purely infinite and simple. Once we have this simplicity result, the rest of the proof that $\mathcal{O}_\infty \otimes A \cong A$ is done by essentially the same methods as those of [Rr3]. (We actually prove, for future use elsewhere, a somewhat more general statement. This disguises the similarity with [Rr3] a little.)

We begin by establishing notation for ultrapowers.

Notation 3.1. We adopt the following notation regarding ultrapowers and associated objects. Let D be a C*-algebra. Then (as in Notation 2.1) $l^\infty(D)$ denotes the C*-algebra of bounded sequences with values in D. For any $d = (d_1, d_2, \dots) \in l^\infty(D)$, the function $n \mapsto \|d_n\|$ is bounded and thus defines a continuous function on the Stone-Čech compactification $\beta\mathbb{N}$ of $\mathbb{N}$. Therefore for each $\omega \in \beta\mathbb{N} - \mathbb{N}$, the limit $\lim_{n\to\omega} \|d_n\|$ exists. In particular, it makes sense to define $\lim_{n\to\omega} d_n = 0$ to mean $\lim_{n\to\omega} \|d_n\| = 0$. We then define a closed ideal $c_\omega(D)$ in $l^\infty(D)$ and the corresponding quotient by

$$c_\omega(D) = \{d \in l^\infty(D) : \lim_{n\to\omega} d_n = 0\} \quad \text{and} \quad D_\omega = l^\infty(D)/c_\omega(D).$$

Denote the quotient map by $\pi_\omega^{(D)} : l^\infty(D) \to D_\omega$. If D is clear from the context, we sometimes write π_ω. Note that $\|\pi_\omega^{(D)}(d)\| = \lim_{n\to\omega} \|d_n\|$.

We will regard D as a subalgebra of D_ω via the diagonal embedding of D in $l^\infty(D)$.

If A is a subalgebra of a C*-algebra B, we denote by $A' \cap B$ the relative commutant of A in B. In particular, with the identification above, we denote by $D' \cap D_\omega$ the relative commutant of D in D_ω.

Remark 3.2. If F is a finite dimensional C*-algebra, then $\pi_\omega^{(F)}$ is an isomorphism. More generally, if F is finite dimensional and D is arbitrary, then $\pi_\omega^{(F\otimes D)}$ defines an isomorphism $(F \otimes D)_\omega \to F \otimes D_\omega$.

The following lemma contains the main part of the proof that $A' \cap A_\omega$ is simple.

Lemma 3.3. Let A be a separable nuclear unital purely infinite simple C*-algebra, and let $\omega \in \beta\mathbb{N} - \mathbb{N}$. Let a, $b \in A' \cap A_\omega$ be selfadjoint with $\mathrm{sp}(b) \subset \mathrm{sp}(a)$. Then there is a nonunitary isometry $s \in A' \cap A_\omega$ such that ss^* commutes with a and $s^*as = b$.

Proof: Scaling both a and b by the same factor, we may assume that $\|a\|$, $\|b\| \le \pi/2$. Let $v = \exp(ia)$, and let $X = \mathrm{sp}(v)$, which is a subset of S^1 intersected with the right halfplane. Let $z \in C(X)$ be the standard generating unitary, $z(\zeta) = \zeta$. Then the assignments

$$z \otimes 1 \mapsto v \quad \text{and} \quad 1 \otimes x \mapsto \pi_\omega(x, x, \dots)$$

define a unital homomorphism $\varphi : C(X) \otimes A \to A_\omega$, and similarly the assignments

$$z \otimes 1 \mapsto \exp(ib) \quad \text{and} \quad 1 \otimes x \mapsto \pi_\omega(x, x, \dots)$$

define a unital homomorphism $\psi : C(X) \otimes A \to A_\omega$.

We prove that φ is injective. (Note that ψ need not be injective.) To see this, note that, since A is simple, $\ker(\varphi)$ must have the form $C_0(U) \otimes A$ for some open subset $U \subset X$. If $U \neq \emptyset$, then there is a nonzero continuous function $f \in C_0(U)$. This gives $0 = \varphi(f \otimes 1) = f(v)$, contradicting the fact that f is a nonzero element of $C(\mathrm{sp}(v))$.

Since $C(X) \otimes A$ is nuclear, there are, by Theorem 0.3, unital completely positive maps V, $W : C(X) \otimes A \to l^\infty(A)$ which lift φ and ψ. Then V has the form $V(x) = (V_1(x), V_2(x), \dots)$ for unital completely positive maps $V_m : C(X) \otimes A \to A$, and similarly $W(x) = (W_1(x), W_2(x), \dots)$ with W_m unital and completely positive. We next want to apply Lemma 1.10, and for this we need information on the injectivity of V_m on finite dimensional subspaces.

Choose a sequence $u_1, u_2, \dots$ of unitaries in A whose linear span is dense, and let $E_n \subset C(X) \otimes A$ be given by

$$E_n = \mathrm{span}\{1, z \otimes 1, z^* \otimes 1, 1 \otimes u_1, 1 \otimes u_1^*, \dots, 1 \otimes u_n, 1 \otimes u_n^*\}.$$

Temporarily fix n and k. For $x \in E_n \otimes M_k \subset C(X) \otimes A \otimes M_k$, we have (using Remark 3.2 and the definition of $c_\omega(A)$ in the first step and injectivity of $\varphi \otimes \mathrm{id}_{M_k}$ in the second)

$$\lim_{m\to\omega} \|(V_m \otimes \mathrm{id}_{M_k})(x)\| = \|(\varphi \otimes \mathrm{id}_{M_k})(x)\| = \|x\|.$$

Since E_n is finite dimensional, it follows that there is a neighborhood U of ω in $\beta\mathbb{N}$ such that for all $m \in U \cap \mathbb{N}$, the map $V_m|_{E_n}$ is invertible; moreover, $\lim_{m\to\omega} \|(V_m|_{E_n})^{-1} \otimes \mathrm{id}_{M_k}\| = 1$. This holds for all n and k.

Using Lemma 1.10, choose positive integers $k(m)$ with $k(0) \le k(1) \le \cdots$ and such that whenever $\tilde{V}$, $\tilde{W} : E_n \to A$ are unital completely positive with $\tilde{V}$ injective and $\|\tilde{V}^{-1} \otimes \mathrm{id}_{M_{k(m)}}\| \le 1 + 1/m$, then there is

a unital completely positive map $T : A \to A$ such that $\|T \circ \tilde{V} - \tilde{W}\| < 2/m$. Choose a decreasing sequence of neighborhoods $U_1 \supset U_2 \supset \cdots$ of ω in $\beta\mathbb{N}$ such that for all $m \in U_n \cap \mathbb{N}$ we have

$$\|(V_m|_{E_n})^{-1} \otimes \mathrm{id}_{M_{k(m)}}\| \leq 1 + 1/m.$$

Replacing U_n by $U_n - \{1, 2, \ldots, n\}$, we may assume that $\mathbb{N} \cap \bigcap_{n+1}^{\infty} U_n = \emptyset$. (Note that we can't have $\bigcap_{n+1}^{\infty} U_n = \{\omega\}$, since ω doesn't have a countable neighborhood base. However, it is certainly true that if $f : \mathbb{N} \to \mathbb{C}$ is a function such that $\lim_{n\to\infty} \sup_{m\in U_n} |f(m)| = 0$, then $\lim_{n\to\omega} f(n) = 0$.) Lemma 1.10 now provides unital completely positive maps $T_m : A \to A$ such that $\|T_m \circ V_m|_{E_n} - W_m|_{E_n}\| \leq 2/m$ for $m \in (U_n - U_{n+1}) \cap \mathbb{N}$. By Proposition 1.7 (combined with the compactness of the closed unit ball of E_n), there are nonunitary isometries $s_m \in A$ for $m \in (U_n - U_{n+1}) \cap \mathbb{N}$ such that

$$\|s_m^* V_m(x) s_m - W_m(x)\| \leq 3\|x\|/m$$

for all $x \in E_n$. Since

$$E_n \subset E_{n+1} \subset \cdots \quad \text{and} \quad \mathbb{N} \cap U_n \cap U_{n+1} \cap \cdots = \emptyset,$$

this estimate in fact holds for all $m \in U_n \cap \mathbb{N}$ and $x \in E_n$.

Define $s = \pi_\omega(s_1, s_2, \cdots)$. Clearly s is an isometry in A_ω. It is not unitary since

$$\|1 - ss^*\| = \lim_{m\to\omega} \|1 - s_m s_m^*\| = 1.$$

For $m \in U_n \cap \mathbb{N}$ we have

$$\|s_m^* u_n s_m - u_n\| \leq \|V_m(1 \otimes u_n) - u_n\| + \|W_m(1 \otimes u_n) - u_n\| + 3/m.$$

Fix n and let $m \to \omega$. The last term on the right certainly converges to 0. The first two terms do so as well because, recalling our identification of A as a subalgebra of A_ω, we have $\pi_\omega(V(1 \otimes u_n)) = \pi_\omega(W(1 \otimes u_n)) = u_n$. Thus $\lim_{m\to\omega} \|s_m^* u_n s_m - u_n\| = 0$ for each fixed n. It follows that $s^* u_n s = u_n$ for all n. Using $z \otimes 1$ in place of $1 \otimes u_n$, we also obtain $s^* v s = \exp(ib)$. Since u_n and $\exp(ib)$ are unitary, Lemma 1.11 implies that ss^* commutes with u_n and v.

Since ss^* commutes with u_n and $s^* u_n s = u_n$, we have

$$s u_n = ss^* u_n s = u_n ss^* s = u_n s.$$

Since $u_1, u_2, \ldots$ span a dense subspace of A, this implies that $s \in A' \cap A_\omega$.

Since ss^* commutes with v, it also commutes with v^*, and it follows that $x \mapsto s^* x s = s^*[ss^* x ss^*]s$ is a homomorphism from the unital C*-algebra generated by v to A_ω. So $s^* f(v) s = f(s^* v s)$ for every continuous function f on $\mathrm{sp}(v)$. Taking $f = -i \log$, we obtain $s^* a s = b$. This completes the proof. ∎

Proposition 3.4. Let A be a separable nuclear unital purely infinite simple C*-algebra, and let $\omega \in \beta\mathbb{N} - \mathbb{N}$. Then $A' \cap A_\omega$ is unital, simple, and purely infinite.

Proof: Obviously $A' \cap A_\omega$ is unital. We show that every nonzero hereditary subalgebra B of $A' \cap A_\omega$ contains a projection $e \neq 1$ which is Murray-von Neumann equivalent to 1. (This clearly implies that $A' \cap A_\omega$ is simple and that every nonzero hereditary subalgebra of $A' \cap A_\omega$ contains an infinite projection.) So choose $c \in B$ selfadjoint with $1 \in \mathrm{sp}(c)$. Apply the previous lemma with $a = c$ and $b = 1$ to obtain a nonunitary isometry $s \in A' \cap A_\omega$ such that $s^* c s = 1$ and the projection $e = ss^*$ commutes with c. By construction, e is Murray-von Neumann equivalent to 1. Furthermore, $ece = s(s^* c s)s^* = e$, and from $ec = ce$ we get $cec = (ece)^2 = e^2 = e$. Therefore $e \in B$. ∎

We note that the proposition can be proved without knowing that the isometry of Lemma 3.3 can be chosen to be nonunitary. One still gets from the proof above that each $A' \cap A_\omega$ is simple and either purely infinite or isomorphic to $\mathbb{C}$. It follows from [AP2] that A has a nontrivial central sequence, that is, a sequence $a \in l^\infty(A)$ such that $\pi_\infty(a) \in A' \cap A_\infty$ but for which there is no sequence $z \in l^\infty(Z(A))$ satisfying $\lim_{n\to\infty} \|a_n - z_n\| = 0$. From this it is possible to deduce that there is at least one $\omega_0 \in \beta\mathbb{N} - \mathbb{N}$ such that $A' \cap A_{\omega_0} \not\cong \mathbb{C}$. One can then show that $A' \cap A_\infty$ contains a unital copy of $\mathcal{O}_\infty$. For every other $\omega \in \beta\mathbb{N} - \mathbb{N}$, the image of this subalgebra in $A' \cap A_\omega$ is nontrivial, so also $A' \cap A_\omega \not\cong \mathbb{C}$.

Definition 3.5. Let A and B be separable unital C*-algebras. An *asymptotically central inclusion* of A in B is a sequence of unital injective homomorphisms $\varphi_n : A \to B$ such that $\|\varphi_n(a)b - b\varphi_n(a)\| \to 0$ for all $a \in A$ and $b \in B$.

The rest of the proof that $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$ was inspired by a talk by Mikael Rørdam. The following lemma is the main remaining part.

Lemma 3.6. Let A be a simple separable unital nuclear C*-algebra which has an asymptotically central inclusion of $\mathcal{O}_2$. Then $A \cong \mathcal{O}_2$.

Proof: We have a unital homomorphism $\varphi : \mathcal{O}_2 \to A$ by assumption, and a unital homomorphism $\psi : A \to \mathcal{O}_2$ by Theorem 2.8. Furthermore, $\psi \circ \varphi$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$ by Proposition 0.7. In the rest of the proof, we show that any two unital endomorphisms of A are approximately unitarily equivalent. In particular, we then have $\varphi \circ \psi$ approximately unitarily equivalent to id_A, so that $A \cong \mathcal{O}_2$ by Lemma 0.6.

First observe that A is purely infinite. Indeed, clearly A is infinite. Moreover, $\mathcal{O}_2$ is approximately divisible in the sense of [BKR], by Proposition 7.7 of [Rr1]. Therefore so is A. So A is purely infinite by Theorem 1.4 (a) of [BKR].

Now let $\gamma : A \to A$ be a unital endomorphism; we show that γ is approximately unitarily equivalent to id_A. Certainly γ is nuclear, unital, and completely positive, so by Proposition 1.7 there are isometries $v_n \in A$ such that $\lim_{n\to\infty} v_n^* a v_n = \gamma(a)$ for all $a \in A$. Choose any $\omega \in \beta\mathbb{N} - \mathbb{N}$, and set $v = \pi_\omega^{(A)}(v_1, v_2, \dots)$, which is an isometry in A_ω. Regarding A as a subalgebra of A_ω as in Notation 3.1, we have $v^*av = \gamma(a)$ for all $a \in A$. It now follows from Lemma 1.11 that vv^* commutes with every unitary in A, whence $vv^* \in A' \cap A_\omega$.

Let $F_1 \subset F_2 \subset \cdots$ be finite selfadjoint subsets of A whose union is dense in A, and such that $v_n v_n^* \in F_n$. Let s_1 and s_2 be the standard generating isometries of $\mathcal{O}_2$. The existence of an asymptotically central inclusion of $\mathcal{O}_2$ in A provides unital homomorphisms $\sigma_n : \mathcal{O}_2 \to A$ such that $\|\sigma_n(x)a - a\sigma_n(x)\| < \frac{1}{n}$ for $a \in F_n$ and x in the generating set $\{s_1, s_1^*, s_2, s_2^*\}$ of $\mathcal{O}_2$. The definition $\sigma(x) = \pi_\omega^{(A)}(\sigma_1(x), \sigma_2(x), \dots)$ yields a unital homomorphism $\sigma : \mathcal{O}_2 \to A' \cap A_\omega$ whose range also commutes with vv^*. We then calculate in $K_0(A' \cap A_\omega)$:

$$[vv^*] = [\sigma(s_1)vv^*\sigma(s_1)^* + \sigma(s_2)vv^*\sigma(s_2)^*] = 2[vv^*],$$

so that $[vv^*] = 0$ in $K_0(A' \cap A_\omega)$. Similarly $[1] = 0$ in $K_0(A' \cap A_\omega)$. Since $A' \cap A_\omega$ is purely infinite and simple, it follows from Theorem 1.4 and Proposition 1.5 of [Cn2] that there is $w \in A' \cap A_\omega$ such that $w^*w = 1$ and $ww^* = vv^*$. Then $u = w^*v$ is a unitary in A_ω; moreover, for $a \in A$ we have

$$u^*au = v^*waw^*v = v^*av = \gamma(a).$$

To show that γ is approximately unitarily equivalent to id_A, let $F \subset A$ be finite, with $\|a\| \leq 1$ for $a \in F$, and let and $\varepsilon > 0$. Choose $(b_1, b_2, \dots) \in l^\infty(A)$ such that $\pi_\omega^{(A)}(b_1, b_2, \dots) = u$. Without loss of generality, $\|b_n\| \leq 1$ for all n. There is a neighborhood U of ω in $\beta\mathbb{N}$ such that, for every $n \in U \cap \mathbb{N}$, the unitary $u_n = b_n(b_n^*b_n)^{-1/2}$ satisfies $\|u_n - b_n\| \leq \frac{\varepsilon}{3}$, and also $\|b_n^*ab_n - \gamma(a)\| < \frac{\varepsilon}{3}$ for all $a \in F$. Choose one such n; then $\|u_n^*au_n - \gamma(a)\| < \varepsilon$ for all $a \in F$. ∎

Theorem 3.7. Let A be a simple separable unital nuclear C*-algebra. Then $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$.

Proof: Let $B = \bigotimes_1^\infty \mathcal{O}_2$, which we think of as $\varinjlim \bigotimes_1^n \mathcal{O}_2$. Obviously there is an asymptotically central inclusion of $\mathcal{O}_2$ in B. So there is also an asymptotically central inclusion of $\mathcal{O}_2$ in $B \otimes A$. The previous lemma therefore implies that $B \cong \mathcal{O}_2$ and $B \otimes A \cong \mathcal{O}_2$. So $\mathcal{O}_2 \otimes A \cong \mathcal{O}_2$. ∎

Remark 3.8. It is actually possible to get this far (except for Theorem 1.15, which we haven't used yet) with only a unital (necessarily injective) homomorphism $\mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$. Then one would get $\mathcal{O}_2 \otimes \mathcal{O}_2 \cong \mathcal{O}_2$ as a corollary to the previous theorem. However, there doesn't seem to be a way to get such a homomorphism which is simpler than going through much of Rørdam's proof of the isomorphism.

We now turn to the proof that $\mathcal{O}_\infty \otimes A \cong A$. For use elsewhere, we prove a somewhat more general statement, in which $\mathcal{O}_\infty$ is replaced by a subalgebra of $B \subset A' \cap A_\omega$. We will eventually take $B = \bigotimes_1^\infty \mathcal{O}_\infty$, but for now we merely assume that it is separable, that it contains the unit of $A' \cap A_\omega$, and that the two obvious maps from B to $B \otimes_{\min} B$ are approximately unitarily equivalent.

Lemma 3.9. (Compare with Propositions 2.7 and 2.8 of [EfR].) Let B be a separable unital C*-algebra. Suppose that the two maps $\alpha,\ \beta : B \to B \otimes_{\min} B$, given by

$$\alpha(b) = b \otimes 1 \quad \text{and} \quad \beta(b) = 1 \otimes b,$$

are approximately unitarily equivalent. Then B is simple and nuclear.

Proof: We prove nuclearity first. This argument is taken from [EfR]. By hypothesis there is a sequence $u_1, u_2, \ldots$ of unitaries in $B \otimes_{\min} B$ such that $\lim_{n\to\infty} u_n(b\otimes 1)u_n^* = 1 \otimes b$ for all $b \in B$. Choose c_n in the algebraic tensor product $B \otimes_{\mathrm{alg}} B$ with $\|c_n\| \leq 1$ and $\lim_{n\to\infty} \|c_n - u_n\| = 0$. Then also $\lim_{n\to\infty} c_n(b \otimes 1)c_n^* = 1 \otimes b$ for all $b \in B$. Choose any state ω on B, and define $T_n : B \to B$ by

$$T_n(b) = (\omega \otimes \mathrm{id}_B)(c_n(b \otimes 1)c_n^*).$$

Note that $\omega \otimes \mathrm{id}_B : B \otimes_{\min} B \to B$ is well defined, unital, and completely positive. (See, for example, Proposition IV.4.23 (i) of [Tk].) Since $\|c_n\| \leq 1$, the maps T_n are thus completely positive contractions.

For fixed n, write $c_n = \sum_{j=1}^m x_j \otimes y_j$ with $x_j,\ y_j \in B$. Then

$$T_n(b) = \sum_{j,\,k=1}^{m} \omega(x_j b x_k^*) y_j y_k^*,$$

so that T_n has finite rank (at most m^2). We further have

$$\|T_n(b) - b\| = \|(\omega \otimes \mathrm{id}_B)(c_n(b \otimes 1)c_n^*) - (\omega \otimes \mathrm{id}_B)(1 \otimes b)\| \leq \|c_n(b \otimes 1)c_n^* - 1 \otimes b\|,$$

which converges to 0 as $n \to \infty$. Thus, we have shown that id_B is a pointwise norm limit of completely positive contractions. So B is nuclear. (It is not necessary to require that the approximating maps be unital. See the comment after Theorem 0.2.)

Now we prove simplicity. Suppose B has a nontrivial ideal J. As in the proof of Proposition 2.7 of [EfR], if $b \in J$ then $b \otimes 1 \in J \otimes_{\min} B$ but $b \otimes 1 \notin B \otimes_{\min} J$. However, with u_n as in the first paragraph of the proof, we have

$$u_n^*(1 \otimes b)u_n \in B \otimes_{\min} J \quad \text{and} \quad \lim_{n\to\infty} u_n^*(1 \otimes b)u_n = b \otimes 1,$$

which implies $b \otimes 1 \in B \otimes_{\min} J$. This contradiction shows that B is simple. ∎

We can now write simply $\otimes$ instead of $\otimes_{\min}$ for tensor products involving B.

Lemma 3.10. Let A, B, and C be separable unital C*-algebras, and let $\omega \in \beta\mathbb{N} - \mathbb{N}$. Let $S(b) = (S_1(b), S_2(b), \ldots)$ and $T(c) = (T_1(c), T_2(c), \ldots)$ define unital completely positive maps from B and C respectively to $l^\infty(A)$ such that $\pi_\omega \circ S$ and $\pi_\omega \circ T$ are unital homomorphisms whose images lie in $A' \cap A_\omega$. For any finite subsets $F \subset A$, $G \subset B$, and $H \subset C$, and any k and $\varepsilon > 0$, there is a neighborhood U of ω such that for every $n \in U \cap \mathbb{N}$, we have

$$\|T_n(c)S_k(b) - S_k(b)T_n(c)\| < \varepsilon, \quad \|T_n(c)a - aT_n(c)\| < \varepsilon, \quad \text{and} \quad \|T_n(c_1c_2) - T_n(c_1)T_n(c_2)\| < \varepsilon,$$

for all $a \in F$, $b \in G$, and $c,\ c_1,\ c_2 \in H$.

Proof: This is immediate. ∎

The following lemma is closely related to Lemmas 2 and 3 of [Rr3].

Lemma 3.11. Let A be a separable unital C*-algebra, let $\omega \in \beta\mathbb{N} - \mathbb{N}$, and let $B \subset A' \cap A_\omega$ be a unital subalgebra which satisfies the hypotheses of Lemma 3.9. Then there is a unital homomorphism $\varphi : B \otimes A \to A$ such that the map $a \mapsto \varphi(1 \otimes a)$ is approximately unitarily equivalent to id_A.

Proof: Since B is nuclear (by Lemma 3.9), its inclusion in A_ω lifts to a unital completely positive map $Q : B \to l^\infty(A)$. We write this map as $Q(b) = (Q_1(b), Q_2(b), \ldots)$ for unital completely positive maps $Q_n : B \to A$.

Choose finite selfadjoint subsets

$$F_1 \subset F_2 \subset \cdots \subset A \quad \text{and} \quad G_1 \subset G_2 \subset \cdots \subset B$$

whose unions are dense in A and B. Using the hypothesis on B, choose unitaries $u_k \in B \otimes B$ such that $\|u_k(b \otimes 1)u_k^* - 1 \otimes b\| < 2^{-k}$ for $b \in G_k$. Choose $b_k \in B \otimes B$ in the algebraic tensor product and so close to u_k

that the unitary $z = b_k (b_k^* b_k)^{-1/2}$ satisfies $\|z(b \otimes 1)z^* - 1 \otimes b\| < 2 \cdot 2^{-k}$ for $b \in G_k$. Write $b_k = \sum_{j=1}^{l} d_j^{(1)} \otimes d_j^{(2)}$ (suppressing the dependence on k on the right). There are then a finite set $G'_k \subset B$ (containing G_k and all $d_j^{(i)}$) and $\varepsilon_k > 0$ such that whenever $S, T : B \to A$ are unital completely positive maps such that

$$\|S(bc) - S(b)S(c)\| < \varepsilon_k, \quad \|T(bc) - T(b)T(c)\| < \varepsilon_k, \quad \|S(b)T(c) - T(c)S(b)\| < \varepsilon_k,$$

$$\|S(b)a - aS(b)\| < \varepsilon_k, \quad \text{and} \quad \|T(b)a - aT(b)\| < \varepsilon_k$$

for $a \in F_k$ and $b, c \in G'_k$, then there is a unitary $v \in A$ (close to $\sum_{j=1}^{l} S(d_j^{(1)})T(d_j^{(2)})$) such that

$$\|vS(b)v^* - T(b)\| < 3 \cdot 2^{-k} \quad \text{and} \quad \|vav^* - a\| < 2^{-k}$$

for $a \in F_k$ and $b \in G_k$. Constructing the G'_k and ε_k in order, we may assume that $G'_k \subset G'_{k+1}$ and $\varepsilon_k > \varepsilon_{k+1}$ for all k, and that $\varepsilon_k \to 0$.

Now use the previous lemma to choose inductively $n(1) < n(2) < \cdots$ such that

$$\|Q_{n(1)}(bc) - Q_{n(1)}(b)Q_{n(1)}(c)\| < \varepsilon_1 \quad \text{and} \quad \|Q_{n(1)}(b)a - aQ_{n(1)}(b)\| < \varepsilon_1$$

for $a \in F_1$ and $b, c \in G'_1$, and

$$\|Q_{n(k+1)}(bc) - Q_{n(k+1)}(b)Q_{n(k+1)}(c)\| < \varepsilon_{k+1}, \quad \|Q_{n(k+1)}(b)a - aQ_{n(k+1)}(b)\| < \varepsilon_{k+1},$$

and

$$\|Q_{n(k+1)}(b)Q_{n(k)}(c) - Q_{n(k)}(c)Q_{n(k+1)}(b)\| < \varepsilon_k$$

for $a \in F_k$ and $b, c \in G'_{k+1}$. The previous paragraph gives a unitary $v_k \in A$ such that

$$\|v_k Q_{n(k+1)}(b) v_k^* - Q_{n(k)}(b)\| < 3 \cdot 2^{-k} \quad \text{and} \quad \|v_k a v_k^* - a\| < 2^{-k}$$

for $a \in F_k$ and $b \in G_k$.

Define a unitary $w_n \in A$ by $w_k = v_1 v_2 \cdots v_{k-1}$. Since $\sum_{k=1}^{\infty} 2^{-k} < \infty$ and $\bigcup_{n=1}^{\infty} F_k$ is dense in A, one checks that $\alpha(a) = \lim_{k\to\infty} w_k a w_k^*$ exists for all $a \in A$. Clearly α is a unital homomorphism from A to A which is approximately unitarily equivalent to id_A. Also,

$$\|w_{k+1} Q_{n(k+1)}(b) w_{k+1}^* - w_k Q_{n(k)}(b) w_k^*\| < 3 \cdot 2^{-k}$$

for $b \in G_k$, so $\beta(b) = \lim_{n\to\infty} w_k Q_{n(k)}(b) w_k^*$ exists for $b \in B$. Clearly $\beta(bc) = \beta(b)\beta(c)$ for $b, c \in \bigcup_{n=1}^{\infty} G'_k$, so β is a unital homomorphism from B to A. Moreover, for $a \in F_k$ and $b \in G_k$, we have

$$\|(w_k a w_k^*)(w_k Q_{n(k)}(b) w_k^*) - (w_k Q_{n(k)}(b) w_k^*)(w_k a w_k^*)\| = \|aQ_{n(k)}(b) - Q_{n(k)}(b)a\| < \varepsilon_k,$$

and $\lim_{k\to\infty} \varepsilon_k = 0$. It follows that the ranges of α and β commute. Since $B \otimes_{\max} A = B \otimes A$, the desired homomorphism $\varphi : B \otimes A \to A$ can be defined by the formula $\varphi(b \otimes a) = \alpha(a)\beta(b)$. ∎

Proposition 3.12. Under the hypotheses of Lemma 3.11, we have $B \otimes A \cong A$.

Proof: Since B is nuclear, we write $\otimes$ rather than $\otimes_{\min}$ for tensor products involving B.

We follow the proof of the theorem following Lemma 3 in [Rr3]. Let $\beta : B \otimes A \to A$ be the homomorphism of Lemma 3.11, and let $\alpha : A \to B \otimes A$ be the homomorphism $\alpha(a) = 1 \otimes a$. We know from Lemma 3.11 that $\beta \circ \alpha$ is approximately unitarily equivalent to id_A. We show that $\alpha \circ \beta$ is approximately unitarily equivalent to $\mathrm{id}_{B \otimes A}$. By Lemma 0.6, this will imply that $B \otimes A \cong A$.

By the definition of approximate unitary equivalence, there is a sequence $w_1, w_2, \ldots$ of unitaries in A such that

$$\lim_{n\to\infty} \|w_n \beta(\alpha(a)) w_n^* - a\| = 0$$

for all $a \in A$. The hypothesis on B provides a sequence $v_1, v_2, \ldots$ of unitaries in $B \otimes B$ such that

$$\lim_{n\to\infty} \|v_n (x \otimes 1) v_n^* - 1 \otimes x\| = 0$$

for all $x \in B$. Note that for $x, y \in B$, the elements $\alpha(\beta(x \otimes 1))$ and $y \otimes 1$ of $B \otimes A$ commute. Therefore there is a homomorphism $\sigma : B \otimes B \to B \otimes A$ satisfying

$$\sigma(x \otimes 1) = \alpha(\beta(x \otimes 1)) \quad \text{and} \quad \sigma(1 \otimes y) = y \otimes 1.$$

Define $u_n = (1 \otimes w_n)\sigma(v_n)$. Using the fact that $\sigma(v_n)$ commutes with $\alpha(\beta(1 \otimes a))$ for $a \in A$ and $1 \otimes w_n$ commutes with $x \otimes 1 = \lim_{n\to\infty} \sigma(v_n)\alpha(\beta(x \otimes 1))\sigma(v_n)^*$ for $x \in B$, we obtain as in [Rr3] $\lim_{n\to\infty} u_n \alpha(\beta(c))u_n^* = c$ for all $c \in B \otimes A$. ∎

Lemma 3.13. Let A be a separable unital C*-algebra, let $\omega \in \beta\mathbb{N} - \mathbb{N}$, and let $B \subset A' \cap A_\omega$ be a separable nuclear unital subalgebra. Let C be a separable nuclear unital C*-algebra, and let $\varphi : C \to A' \cap A_\omega$ be a unital homomorphism. Then there is a unital homomorphism $\psi : C \to A' \cap A_\omega$ whose image lies in the relative commutant of B.

Proof: Since B and C are nuclear, there are unital completely positive maps $S : B \to l^\infty(A)$ and $T : C \to l^\infty(A)$ such that $\pi_\omega \circ S$ is the inclusion of B and $\pi_\omega \circ T = \varphi$. Write $S(b) = (S_1(b), S_2(b), \dots)$ and $T(c) = (T_1(c), T_2(c), \dots)$. Choose finite subsets

$$F_1 \subset F_2 \subset \cdots \subset A, \quad G_1 \subset G_2 \subset \cdots \subset B, \quad \text{and} \quad H_1 \subset H_2 \subset \cdots \subset C$$

whose unions are dense. Using Lemma 3.10, choose $n(1)\, n(2), \dots \in \mathbb{N}$ such that

$$\|T_{n(k)}(c)S_k(b) - S_k(b)T_{n(k)}(c)\| < \frac{1}{k}, \quad \|T_{n(k)}(c)a - aT_{n(k)}(c)\| < \frac{1}{k},$$

and

$$\|T_{n(k)}(c_1c_2) - T_{n(k)}(c_1)T_{n(k)}(c_2)\| < \frac{1}{k}$$

for all $a \in F_k$, $b \in G_k$, and c, c_1, $c_2 \in H_k$. Then define $\psi(c) = \pi_\omega(T_{n(1)}(c), T_{n(2)}(c), \dots)$. ∎

Theorem 3.14. Let A be a separable nuclear unital purely infinite simple C*-algebra. Then $\mathcal{O}_\infty \otimes A \cong A$.

Proof: Let $B = \bigotimes_1^\infty \mathcal{O}_\infty$, which we think of as the direct limit over n of $B_n = \bigotimes_1^n \mathcal{O}_\infty$, with maps $b \mapsto b \otimes 1$. We apply Proposition 3.12 with this B. Clearly B is separable, unital, and nuclear. We need to check two more conditions.

First, we must embed B as a unital subalgebra of $A' \cap A_\omega$. Now $A' \cap A_\omega$ is purely infinite simple by Proposition 3.4, so certainly contains a unital copy of $\mathcal{O}_\infty$. Using the previous lemma and induction, we obtain unital homomorphisms $\varphi_n : B_n \to A' \cap A_\omega$ such that $\varphi_{n+1}(b \otimes 1) = \varphi_n(b)$ for $b \in B_n$. Taking direct limits gives a unital homomorphism $\varphi : B \to A' \cap A_\omega$. This homomorphism is injective because B is simple.

The second condition to check is that the two maps $\alpha(b) = b \otimes 1$ and $\beta(b) = 1 \otimes b$, from B to $B \otimes B$, are approximately unitarily equivalent.

For $F \subset \mathcal{O}_\infty$, let $F^{(n)} \subset B_n$ be the set of all $1 \otimes \cdots \otimes 1 \otimes b \otimes 1 \otimes \cdots \otimes 1$, with $b \in F$, and where b is in the k-th tensor factor for some k with $1 \leq k \leq n$. Also let $\psi_n : B_n \to B$ be the inclusion. It suffices to show that for each finite $F \subset \mathcal{O}_\infty$, each n, and each $\varepsilon > 0$, there exists a unitary $v \in B \otimes B$ such that $\|v(\psi_n(x) \otimes 1)v^* - 1 \otimes \psi_n(x)\| < \varepsilon$ for all $x \in F^{(n)}$.

Now clearly B has an asymptotically central inclusion of $\mathcal{O}_\infty$. It follows that B is approximately divisible in the sense of [BKR]. Since B is simple and infinite, it is purely infinite by Theorem 1.4 (a) of [BKR]. So Theorem 3.3 of [LP2] implies that the maps from $\mathcal{O}_\infty$ to $B \otimes B$, given by $x \mapsto \psi_1(x) \otimes 1$ and $x \mapsto 1 \otimes \psi_1(x)$, are approximately unitarily equivalent. Approximating unitaries in B by ones in the terms of the direct system, we find that for F and ε as above there is N and $u \in B_N \otimes B_N$ such that, with 1_m denoting the identity of $\bigotimes_1^m \mathcal{O}_\infty$, we have

$$\|u[(b \otimes 1_{N-1}) \otimes 1_N]u^* - 1_N \otimes (b \otimes 1_{N-1})\| < \varepsilon$$

for all $b \in F$. Taking the tensor product of this with itself n times, we find that $v_0 = u \otimes \cdots \otimes u \in \bigotimes_1^n (B_N \otimes B_N)$ satisfies

$$\|v_0(1 \otimes \cdots \otimes 1 \otimes [(b \otimes 1_{N-1}) \otimes 1_N] \otimes 1 \otimes \cdots \otimes 1)v_0^* - 1 \otimes \cdots \otimes 1 \otimes [1_N \otimes (b \otimes 1_{N-1})] \otimes 1 \otimes \cdots \otimes 1\| < \varepsilon$$

for all $b \in F$ and with $(b \otimes 1_{N-1}) \otimes 1_N$ in any of the n factors $B_N \otimes B_N$. Rearranging this using associativity of the tensor product, we obtain a unitary $v \in B_{Nn} \otimes B_{Nn}$ such that

$$\|v[(x \otimes 1_{(N-1)n}) \otimes 1_{Nn}]v^* - 1_{Nn} \otimes (x \otimes 1_{(N-1)n})\| < \varepsilon$$

for all $x \in F^{(n)}$. Embedding B_{Nn} in B completes the proof that α and β are approximately unitarily equivalent.

We now apply Proposition 3.12, obtaining $B \otimes A \cong A$. Taking in particular $A = \mathcal{O}_\infty$, we obtain $\mathcal{O}_\infty \cong B \otimes \mathcal{O}_\infty$. But clearly $B \otimes \mathcal{O}_\infty \cong B$. So we can replace B by $\mathcal{O}_\infty$, getting $\mathcal{O}_\infty \otimes A \cong A$. ∎

4. Exact continuous fields

This section consists of various preparatory results on (exact) continuous fields of C*-algebras which will be used in the next section to obtain continuous embeddings of continuous fields in $\mathcal{O}_2$. We start with several general results, on tensor products, pullbacks, and representations, and then go on to apply the results of Section 1 to continuous fields whose section algebras are exact. We obtain a discrete version of continuous embedding: Fibers over nearby points have embeddings in $\mathcal{O}_2$ which are close in a suitable sense. In the next section, we show how to make continuously varying choices of the embeddings.

Continuous fields of C*-algebras are taken to be as defined in Chapter 10 of [Dx]. For notation, if A is a continuous field over X, then we denote by $A(x)$ the fiber over $x \in X$ and by $\Gamma(A)$ the set of all continuous sections of A. We briefly recall the axioms for $\Gamma(A)$ ([Dx], 10.1.2 and 10.3.1):

(1) Each $A(x)$ is a C*-algebra.

(2) The section space $\Gamma(A)$ is a subspace of the set-theoretic product $\prod_{x \in X} A(x)$ which is closed under addition, scalar multiplication, multiplication, and adjoint.

(3) The set $\{a(x) : a \in \Gamma(A)\}$ is dense in $A(x)$ for all x. (Proposition 10.1.10 of [Dx] shows that, in the presence of the other axioms, this is equivalent to requiring that $\{a(x) : a \in \Gamma(A)\} = A(x)$ for all x.)

(4) For every $a \in \Gamma(A)$ the function $x \mapsto \|a(x)\|$ is continuous.

(5) The section space $\Gamma(A)$ is closed under local uniform approximation. That is, if a is a section, and if for every $x_0 \in X$ and $\varepsilon > 0$ there exists a neighborhood U of x_0 and a continuous section b such that $\|a(x) - b(x)\| < \varepsilon$ for $x \in U$, then a is continuous.

It follows from Axiom (5) that the pointwise scalar product of a continuous function on X and a continuous section of A is again a continuous section of A. (See Proposition 10.1.9 of [Dx].)

More general bundles, in which (4) is weakened to merely require that the norms of sections be upper semicontinuous, are in some ways more natural. (See [HK] for a general discussion, mostly in the context of Banach spaces.) However, the results of this and the next section have no possibility of being true for them.

We denote by ev_x the evaluation map $a \mapsto a(x)$ from $\Gamma(A)$ to $A(x)$. We say that A is *unital* if each $A(x)$ is unital and the section $x \mapsto 1_{A(x)}$ is continuous. Note that it is possible to have every $A(x)$ unital but the section $x \mapsto 1_{A(x)}$ discontinuous.

We start with several general results on continuous fields.

Proposition 4.1. Let X be a compact Hausdorff space, let A be a continuous field of C*-algebras over X, and let B be a nuclear C*-algebra. Then there is a continuous field $A \otimes B$ of C*-algebras over X, such that $(A \otimes B)(x) = A(x) \otimes B$ and $\Gamma(A \otimes B) = \Gamma(A) \otimes B$.

Proof: This is (i) implies (v) of Theorem 3.2 of [KW]. ∎

The cases we need are $B = M_n$ and $B = K$, which can be handled a little more directly.

Lemma 4.2. Let X be a topological space, and let A be a continuous field of C*-algebras over X. Then there is a continuous field $A^\dagger$ of C*-algebras over X such that $A^\dagger(x)$ is the unitization $A(x)^\dagger$ for every $x \in X$, and such that the continuous sections of $A^\dagger$ are the sections of the form $a(x) = a_0(x) + \lambda(x) \cdot 1_{A(x)^\dagger}$ for a_0 a continuous section of A and $\lambda : X \to \mathbb{C}$ continuous.

Proof: We define $A^\dagger(x) = A(x)^\dagger$ for each x. (Recall that we add a new identity to $A(x)$ even if it already has one.) We take the continuous sections of $A^\dagger$ to be as in the statement. Axioms (1), (2), and (3) are obvious. Axiom (5) is easily checked using the fact that $A(x)^\dagger = A(x) \oplus \mathbb{C}$ as a Banach space. This leaves axiom (4).

Considering $a(x)^* a(x)$, we see that it suffices to consider sections $a(x) = a_0(x) + \lambda(x) \cdot 1_{A(x)^\dagger}$ with $a(x) \geq 0$ for all x. In this case, the corresponding function λ is nonnegative and a_0 is selfadjoint. Let $f : \mathbb{C} \to \mathbb{R}$ be $f(t) = \max(\mathrm{Re}(t), 0)$. By Proposition 10.3.3 of [Dx], the section $x \mapsto f(a_0(x))$ is continuous. Now

$$\|a(x)\| = \lambda(x) + \sup \mathrm{sp}(a_0(x)) = \lambda(x) + \|f(a_0(x))\|,$$

which is continuous. ∎

Lemma 4.3. Let $f : X \to Y$ be a continuous map of topological spaces, and let A be a continuous field over Y. Then there is a continuous field $f^*(A)$ over X such that $f^*(A)(x) = A(f(x))$ for all $x \in X$, and such that the continuous sections of $f^*(A)$ are the locally uniform limits (in the sense implicit in Axiom (5)) of sections of the form $x \mapsto a(f(x))$ for $a \in \Gamma(A)$.

Proof: It is immediate from the corresponding axioms for A that $f^*(A)$ satisfies axioms (1) and (3). Axiom (5) for $f^*(A)$ follows from the construction of $\Gamma(f^*(A))$ as the set of locally uniform limits of a set of sections. Axioms (2) and (4) hold for the set of sections of the form $x \mapsto a(f(x))$ for $a \in \Gamma(A)$, and are preserved under passage to locally uniform limits, so also hold for $\Gamma(f^*(A))$. ∎

Lemma 4.4. Let X be a locally compact Hausdorff space, let $U \subset X$ be open, and let A_0 be a unital continuous field of C*-algebras over U. Then there is a unital continuous field A of C*-algebras over X such that $A(x) = A_0(x)$ for $x \in U$ and $A(x) = \mathbb{C}$ for $x \notin U$, and such that a section a is continuous if and only if there is a continuous function $\lambda : X \to \mathbb{C}$ and a continuous section a_0 of A_0 for which $x \mapsto \|a_0(x)\|$ vanishes at infinity on U, such that $a(x) = \lambda(x) \cdot 1_{A(x)}$ for $x \notin U$ and $a(x) = \lambda(x) \cdot 1_{A(x)} + a_0(x)$ for $x \in U$.

Proof: One checks directly that the given set of sections satisfies the definition of a unital continuous field of C*-algebras. ∎

The following definition and lemma do not work very well for more general bundles (for which the norm of a section is only required to be upper semicontinuous).

Definition 4.5. Let X be a topological space, and let A be a continuous field of C*-algebras over X. A *representation* of A in a C*-algebra D is a family $\varphi = (\varphi_x)_{x \in X}$ of homomorphisms $\varphi_x : A(x) \to D$ which is continuous in the following sense: for every continuous section a of A, the function $x \mapsto \varphi_x(a(x))$ is continuous from X to D. The representation is called *injective* if every φ_x is injective.

In this terminology, a continuous field is Hilbert continuous (Definition 3.3 of [Rf]) if it has an injective representation in some $L(H)$.

Lemma 4.6. Let X be a topological space, let A be a continuous field of C*-algebras over X, and let $\varphi = (\varphi_x)_{x \in X}$ be a representation of A in some C*-algebra D. Suppose φ_x is injective for every x is some dense subset S of X. Then φ_x is injective for every $x \in X$.

Proof: Let $x \in X$, and suppose φ_x is not injective. Choose an element $a \in \ker(\varphi_x)$ with $\|a\| = 1$. By Proposition 10.1.10 of [Dx], there is a continuous section b of A such that $b(x) = a$. By continuity of $x \mapsto \|b(x)\|$ and $x \mapsto \varphi_x(b(x))$, there is a neighborhood U of x such that $\|b(y)\| > 3/4$ and $\|\varphi_y(b(y))\| < 1/4$ for $y \in U$. Choose $y \in U \cap S$. Then $b(y)$ is an element of $A(y)$ such that $\|\varphi_y(b(y))\| < \|b(y)\|$, so φ_y is not injective. This contradiction proves the lemma. ∎

We will work with continuous fields satisfying the exactness conditions given in the following theorem, essentially in [KW].

Theorem 4.7. Let X be a compact metric space, and let A be a continuous field of C*-algebras over X, with $\Gamma(A)$ separable. Then the following conditions are equivalent:

(1) The section algebra $\Gamma(A)$ is an exact C*-algebra.

(2) Every fiber $A(x)$ is an exact C*-algebra, and the identity maps of the fibers

$$\mathrm{id}_{A(x)} : A(x) \to \Gamma(A)/\{a \in \Gamma(A) : \mathrm{ev}_x(a) = 0\} = A(x)$$

are *locally liftable,* that is, for every finite dimensional operator system $E \subset A(x)$ the inclusion of E in $A(x)$ has a unital completely positive lifting to a map from E to $\Gamma(A)$.

(3) Every fiber $A(x)$ is an exact C*-algebra, and for every C*-algebra B, the tensor product bundle $A \otimes_{\min} B$ (as described in the introduction to [KW]) is a continuous field.

(4) Every fiber $A(x)$ is an exact C*-algebra, and for a separable infinite dimensional Hilbert space H, the tensor product bundle $A \otimes_{\min} L(H)$ is a continuous field.

Proof: Note that if $\Gamma(A)$ is exact, then the fibers $A(x)$, being quotients of $\Gamma(A)$, are exact by Proposition 7.1 (ii) of [Kr4]. So the equivalence of the first three conditions (and some others) is Theorem 4.6 of [KW]. That

(3) implies (4) is immediate. The proof that (4) implies (2) is derived from Theorem 3.2 of [EH] in the same way that Lemma 2.4 is derived from Theorem 3.4 of [EH]. ∎

Lemma 4.8. Let X be a compact metric space, and let A be a continuous field of C*-algebras over X, with $\Gamma(A)$ separable. Let $A^{\dagger}$ be the unitization (as in Lemma 4.2). Then $\Gamma(A)$ is exact if and only if $\Gamma(A^{\dagger})$ is exact.

Proof: There is a split short exact sequence

$$0 \longrightarrow \Gamma(A) \longrightarrow \Gamma(A^{\dagger}) \longrightarrow C(X) \longrightarrow 0.$$

The result therefore follows from Proposition 7.1 (vi) of [Kr4]. ∎

Proposition 4.9. Let X be a compact metric space, and let A be a continuous field of C*-algebras over X with nuclear fibers $A(x)$, and with $\Gamma(A)$ separable. Then A satisfies the conditions of the previous theorem.

Proof: By Theorem 0.3, the maps $\mathrm{id}_{A(x)}$ are in fact liftable, so condition (2) of Theorem 4.7 holds. ∎

The rest of this section is devoted to the proof that if $\Gamma(A)$ is separable and exact, then nearby fibers have nearby embeddings in $\mathcal{O}_2$.

Lemma 4.10. Let X be a compact metric space, let A be a unital continuous field of C*-algebras over X such that $\Gamma(A)$ is separable and exact, and let $a_1, \ldots, a_m$ be continuous sections of A. Let $x_0 \in X$. Then for every $\varepsilon > 0$ there exists a neighborhood U of x_0 in X such that for all $x \in U$ there are injective unital homomorphisms $\varphi_x : A(x_0) \to \mathcal{O}_2$ and $\psi_x : A(x) \to \mathcal{O}_2$, and unital completely positive maps $S_x : A(x_0) \to \mathcal{O}_2$ and $T_x : A(x) \to \mathcal{O}_2$, satisfying

$$\|S_x(a_l(x_0)) - \psi_x(a_l(x))\| < \varepsilon \quad \text{and} \quad \|T_x(a_l(x)) - \varphi_x(a_l(x_0))\| < \varepsilon$$

for $1 \leq l \leq m$.

Proof: By considering the real and imaginary parts of the given sections (and using $\varepsilon/2$ in place of ε), we may assume without loss of generality that $a_1, \ldots, a_m$ are selfadjoint. Similarly, we may assume that $\|a_l\| \leq 1$ for all l.

We next reduce to the case in which $1, a_1(x_0), \ldots, a_m(x_0)$ are linearly independent. In the general case, we may number the a_l in such a way that $1, a_1(x_0), \ldots, a_{m_0}(x_0)$ are linearly independent and $a_{m_0+1}(x_0), \ldots, a_m(x_0)$ are linear combinations of $1, a_1(x_0), \ldots, a_{m_0}(x_0)$. Assume the lemma holds for $a_1, \ldots, a_{m_0}$. Set $a_0 = 1$. For $m_0 + 1 \leq l \leq m$, write

$$a_l(x_0) = \sum_{j=0}^{m_0} \alpha_{jl} a_j(x_0),$$

with $\alpha_{jl} \in \mathbb{C}$, and then define $\tilde{a}_l = \sum_{j=0}^{m_0} \alpha_{jl} a_j$. Then the $\tilde{a}_l$ are also continuous sections, and $\tilde{a}_l(x_0) = a_l(x_0)$. Set

$$\alpha = 1 + \max_{m_0+1 \leq l \leq m} \sum_{j=0}^{m_0} |\alpha_{jl}|.$$

Choose U so small that the conclusion of the lemma holds for $a_1, \ldots, a_{m_0}$, with $\varepsilon/(2\alpha)$ in place of ε, and also so small that $\|\tilde{a}_l(x) - a_l(x)\| < \varepsilon/2$ for $x \in U$ and $m_0 + 1 \leq l \leq m$. The resulting homomorphisms φ_x and ψ_x, and unital completely positive maps S_x and T_x, then satisfy

$$\|S_x(a_l(x_0)) - \psi_x(a_l(x))\| < \frac{\varepsilon}{2\alpha} \quad \text{and} \quad \|T_x(a_l(x)) - \varphi_x(a_l(x_0))\| < \frac{\varepsilon}{2\alpha}$$

for $1 \leq l \leq m_0$. Hence, for $m_0 + 1 \leq l \leq m$ we have

$$\|S_x(a_l(x_0)) - \psi_x(a_l(x))\| \leq \sum_{j=0}^{m_0} |\alpha_{jl}| \|S_x(a_j(x_0)) - \psi_x(a_j(x))\| + \|\psi_x\| \|\tilde{a}_l(x) - a_l(x)\| < \varepsilon.$$

Similarly,

$$\|T_x(a_l(x)) - \varphi_x(a_l(x_0))\| \leq \|T_x\| \|a_l(x) - \tilde{a}_l(x)\| + \sum_{j=0}^{m_0} |\alpha_{jl}| \|T_x(a_j(x)) - \varphi_x(a_j(x_0))\| < \varepsilon.$$

This proves the reduction.

We now assume that $1, a_1, \dots, a_m$ are selfadjoint, have norm at most 1, and are linearly independent at x_0. Set $E = \mathrm{span}(1, a_1(x_0), \dots, a_m(x_0))$; then E is a finite dimensional operator system contained in $A(x_0)$. By local liftability (Theorem 4.7 (2)), there is a unital completely positive map $S : E \to \Gamma(A)$ such that $\mathrm{ev}_x \circ S = \mathrm{id}_E$. Let $b_l = S(a_l(x_0))$, so that b_l is a continuous section of A satisfying $b_l(x_0) = a_l(x_0)$. Choose an open set $U_0 \subset X$, containing x_0, such that $\|b_l(x) - a_l(x)\| < \varepsilon/4$ for all $x \in U_0$. Define $S_x^{(0)} : E \to A(x)$ by $S_x^{(0)} = \mathrm{ev}_x \circ S$. Then $S_x^{(0)}$ is unital and completely positive, and $\|S_x^{(0)}(a_l(x_0)) - a_l(x)\| < \varepsilon/4$ for all $x \in U_0$. Use Theorem 2.8 to find an injective unital homomorphism $\psi_x : A(x) \to \mathcal{O}_2$. Since $\mathcal{O}_2$ is nuclear, Proposition 0.4 provides a unital completely positive map $S_x : A(x_0) \to \mathcal{O}_2$ such that $\|S_x(a_l(x_0)) - \psi_x(S_x^{(0)}(a_l(x_0)))\| < \varepsilon/4$. This gives $\|S_x(a_l(x_0)) - \psi_x(a_l(x))\| < \varepsilon/2$, for all $x \in U_0$.

We still need T_x. Choose, using Lemma 1.10, an integer n such that whenever $V : E \to A(x)$ and $W : E \to \mathcal{O}_2$ are two unital completely positive maps such that V is injective and $\|V^{-1} \otimes \mathrm{id}_{M_n}\| \leq 1 + \varepsilon/4$, there is a unital completely positive map $Q : A(x) \to \mathcal{O}_2$ such that $\|Q \circ V - W\| < \varepsilon/2$. We now claim that there is an open set $U_1 \subset X$, containing x_0, such that $S_x^{(0)}$ is injective and $\|(S_x^{(0)})^{-1} \otimes \mathrm{id}_{M_n}\| \leq 1 + \varepsilon/4$ for all $x \in U_1$. Let $\{e_{ij} : 1 \leq i, j \leq n\}$ be a system of matrix units for M_n. Set $N = (m+1)n^2$, and consider the set of N sections

$$\{c_l : 1 \leq l \leq N\} = \{e_{ij} \otimes 1 : 1 \leq i, j \leq n\} \cup \{e_{ij} \otimes a_l : 1 \leq l \leq m, \ 1 \leq i, j \leq n\}$$

of the continuous field $M_n \otimes A$. (See Lemma 4.1.) Note that $c_1(x_0), \dots, c_N(x_0)$ are linearly independent. Define a compact subset $S \subset \mathbb{C}^N$ by

$$S = \{(\lambda_1, \dots, \lambda_N) \in \mathbb{C}^N : \|\lambda_1 c_1(x_0) + \cdots + \lambda_N c_N(x_0)\| = 1\}.$$

If the claim is false, there is a sequence (y_k) of distinct points in X such that $y_k \to x_0$, and such that there are elements $(\lambda_1^{(k)}, \dots, \lambda_N^{(k)}) \in S$ satisfying

$$\|\lambda_1^{(k)} c_1(y_k) + \cdots + \lambda_N^{(k)} c_N(y_k)\| < \frac{1}{1 + \varepsilon/4}.$$

Passing to a subsequence, we may assume that $\lambda_l = \lim_{k\to\infty} \lambda_l^{(k)}$ exists for each l. By the Tietze extension theorem, there are continuous functions f_l on X such that $f_l(y_k) = \lambda_l^{(k)}$ for all k and $f_l(x_0) = \lambda_l$. Then $c = f_1 c_1 + \cdots + f_N c_N$ is a continuous section with $\|c(x_0)\| = 1$ and $\liminf_{x \to x_0} \|c(x)\| \leq \frac{1}{1+\varepsilon/4}$, a contradiction. This proves the claim.

Set $U = U_0 \cap U_1$. Let $x \in U$. Choose (using Theorem 2.8) some injective unital homomorphism $\varphi_x : A(x_0) \to \mathcal{O}_2$. From above, we have $\|S_x^{(0)}(a_l(x_0)) - a_l(x)\| < \varepsilon/4 < \varepsilon/2$. Furthermore, $V = S_x^{(0)} : E \to A(x)$ and $W = \varphi_x|_E : E \to \mathcal{O}_2$ are unital completely positive maps such that V is injective and $\|V^{-1} \otimes \mathrm{id}_{M_n}\| \leq 1 + \varepsilon/4$. Therefore, by the choice of n and Lemma 1.10, there is a unital completely positive map $T_x : A(x) \to \mathcal{O}_2$ such that $\|T_x \circ S_x^{(0)}|_E - \varphi_x|_E\| < \varepsilon/2$. It follows that

$$\|T_x(a_l(x)) - \varphi_x(a_l(x_0))\| \leq \|a_l(x) - S_x^{(0)}(a_l(x_0))\| + \|T_x \circ S_x^{(0)}(a_l(x_0)) - \varphi_x(a_l(x_0))\| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon,$$

as desired. ∎

Proposition 4.11. Let X, A, and $a_1, \dots, a_m$ be as in Lemma 4.10. Assume we are given, for each $x \in X$, an injective unital homomorphism $\iota_x : A(x) \to \mathcal{O}_2$. Define $\rho_0 : X \times X \to [0, \infty)$ by

$$\rho_0(x, y) = \inf_T \max_{1 \leq l \leq m} \|T(a_l(x)) - \iota_y(a_l(y))\|,$$

where the infemum is taken over all unital completely positive maps $T : A(x) \to \mathcal{O}_2$. Then

(1) $\rho_0(x, y)$ does not depend on the choice of the homomorphisms ι_x.
(2) ρ_0 is continuous on $X \times X$.
(3) $\rho_0(x, x) = 0$ and $\rho_0(x, z) \leq \rho_0(x, y) + \rho_0(y, z)$ for x, y, $z \in X$.

Proof: Part (1) is immediate from the fact (Theorem 1.15) that any two injective unital homomorphisms from $A(y)$ to $\mathcal{O}_2$ are approximately unitarily equivalent.

We next prove (3). We get $\rho_0(x,x) = 0$ by taking $T = \iota_x$. For the triangle inequality, let x, y, $z \in X$, and let $\varepsilon > 0$. Without loss of generality $\|a_l(y)\| \leq 1$ for all l. Choose unital completely positive maps $S : A(x) \to \mathcal{O}_2$ and $T : A(y) \to \mathcal{O}_2$ such that

$$\max_{1 \leq l \leq m} \|S(a_l(x)) - \iota_y(a_l(y))\| \leq \rho_0(x,y) + \frac{\varepsilon}{3} \quad \text{and} \quad \max_{1 \leq l \leq m} \|T(a_l(y)) - \iota_z(a_l(z))\| \leq \rho_0(y,z) + \frac{\varepsilon}{3}.$$

Apply Lemma 1.10 with $A = A(y)$, $B_1 = B_2 = \mathcal{O}_2$, $E = \mathrm{span}(1, a_1(y), \ldots, a_m(y))$, $\delta = 0$, $V = \iota_y|_E$ (so that $\|V^{-1}\|_{\mathrm{cb}} = 1$), and $W = T|_E$, to obtain a unital completely positive map $R : \mathcal{O}_2 \to \mathcal{O}_2$ such that $\|R \circ \iota_y|_E - T|_E\| < \varepsilon/3$. Then $R \circ S : A(x) \to \mathcal{O}_2$ is unital and completely positive and satisfies

$$\begin{aligned} &\|(R \circ S)(a_l(x)) - \iota_z(a_l(z))\| \\ &\quad \leq \quad \|R\|\|S(a_l(x)) - \iota_y(a_l(y))\| + \|(R \circ \iota_y)(a_l(y)) - T(a_l(y))\| + \|T(a_l(y)) - \iota_z(a_l(z))\| \\ &\quad < \quad \left(\rho_0(x,y) + \frac{\varepsilon}{3}\right) + \frac{\varepsilon}{3} + \left(\rho_0(y,z) + \frac{\varepsilon}{3}\right) = \rho_0(x,y) + \rho_0(y,z) + \varepsilon. \end{aligned}$$

This shows that $\rho_0(x,z) \leq \rho_0(x,y) + \rho_0(y,z) + \varepsilon$, and we let $\varepsilon \to 0$.

For continuity (part (2)), let x_0, $y_0 \in X$ and let $\varepsilon > 0$. Use the previous lemma to choose neighborhoods U of x_0 and V of y_0 such that for $x \in U$, both $\rho_0(x_0, x)$ and $\rho_0(x, x_0)$ are less than $\varepsilon/2$, and similarly $\rho_0(y_0, y)$, $\rho_0(y, y_0) < \varepsilon/2$ for $y \in V$. Then for $x \in U$ and $y \in V$,

$$\rho_0(x,y) \leq \rho_0(x,x_0) + \rho_0(x_0,y_0) + \rho_0(y_0,y) < \rho_0(x_0,y_0) + \varepsilon,$$

and similarly $\rho_0(x,y) > \rho_0(x_0,y_0) - \varepsilon$. ∎

Remark 4.12. Let X, A, and $a_1, \ldots, a_m$ be as in Lemma 4.10, and suppose that all the fibers of A are nuclear. (In this case, exactness of $\Gamma(A)$ is automatic, by Proposition 4.9.) Then the definition of the function ρ_0 of the previous proposition can be reformulated to look much more like a distance:

$$\rho_0(x,y) = \inf_T \max_{1 \leq l \leq m} \|T(a_l(x)) - a_l(y)\|,$$

where the infemum is taken over all unital completely positive maps $T : A(x) \to A(y)$.

To see this, let $\tilde{\rho}_0(x,y)$ denote the expression on the right hand side. Obviously, $\rho_0(x,y) \leq \tilde{\rho}_0(x,y)$. For the reverse inequality, let $S : A(x) \to \mathcal{O}_2$ satisfy

$$\max_{1 \leq l \leq m} \|S(a_l(x)) - \iota_y(a_l(y))\| < \rho_0(x,y) + \frac{\varepsilon}{2}.$$

Since $A(y)$ is nuclear, there are n and unital completely positive maps $Q_0 : A(y) \to M_n$ and $R : M_n \to A(y)$ such that $\|R(Q_0(a_l(y))) - a_l(y)\| < \frac{\varepsilon}{2}$ for $1 \leq l \leq m$. The Arveson extension theorem (Theorem 6.5 of [Pl]) gives a unital completely positive map $Q : \mathcal{O}_2 \to M_n$ such that $Q|_{\iota_y(A(y))} = Q_0 \circ \iota_y^{-1}$. Set $T = R \circ Q \circ S$, giving

$$\begin{aligned} \tilde{\rho}_0(x,y) &\leq \max_{1 \leq l \leq m} \|T(a_l(x)) - a_l(y)\| \\ &\leq \quad \|R\|\|Q\| \max_{1 \leq l \leq m} \|S(a_l(x)) - \iota_y(a_l(y))\| + \max_{1 \leq l \leq m} \|(R \circ Q \circ \iota_y)(a_l(y)) - a_l(y)\| < \rho_0(x,y) + \varepsilon. \end{aligned}$$

The square root in the following proposition comes from the one in Lemma 1.12. It can't be removed, as follows from Remark 6.11.

Proposition 4.13. Let X and A be as in Lemma 4.10, and let $u_1, \ldots, u_m$ be continuous unitary sections of A. Define $\rho : X \times X \to [0, \infty)$ by

$$\rho(x,y) = \sup_{\varphi, \psi} \inf_{v \in U(\mathcal{O}_2)} \max_{1 \leq l \leq m} \|v\varphi(u_l(x))v^* - \psi(u_l(y))\|,$$

where the supremum runs over the (nonempty) sets of injective unital homomorphisms $\varphi : A(x) \to \mathcal{O}_2$ and $\psi : A(y) \to \mathcal{O}_2$. Then ρ is a continuous pseudometric on X (that is, a metric except that possibly $\rho(x,y)$ could be zero with $x \neq y$). Moreover, if ρ_0 is as in the previous proposition, using $u_1, \ldots, u_m$ in place of $a_1, \ldots, a_m$, then

$$\rho(x,y) \leq 11\sqrt{\max(\rho_0(x,y), \rho_0(y,x))}.$$

Proof: It follows from Theorem 2.8 that every $A(x)$ posesses an injective unital homomorphism to $\mathcal{O}_2$, and from Theorem 1.15 that any two such homomorphisms are approximately unitarily equivalent. We can therefore rewrite the definition of ρ as follows: For each $x \in X$, choose and fix an injective unital homomorphism $\varphi_x : A(x) \to \mathcal{O}_2$. Then

$$\rho(x, y) = \inf_{v \in U(\mathcal{O}_2)} \max_{1 \leq l \leq m} \|v\varphi_x(u_l(x))v^* - \varphi_y(u_l(y))\|.$$

From this formula, it is obvious that ρ is a pseudometric. (Note that $\rho(x, y)$ can be at most 2 for any x and y.)

It remains to prove the estimate

$$\rho(x, y) \leq 11\sqrt{\max(\rho_0(x, y), \rho_0(y, x))}.$$

(Continuity of ρ follows from this estimate and the previous corollary, using the fact that ρ is a pseudometric.) Equivalently, we prove that for all $\varepsilon > 0$, there is $v \in U(\mathcal{O}_2)$ such that

$$\|v\varphi(u_l(x))v^* - \psi(u_l(y))\| < \varepsilon + 11\sqrt{\max(\rho_0(x, y), \rho_0(y, x))}$$

for $1 \leq l \leq m$.

Fix x and y, and let $\varepsilon > 0$. Choose $\varepsilon_0 > 0$ so small that

$$2\varepsilon_0 + 11\sqrt{\max(\rho_0(x, y), \rho_0(y, x)) + 3\varepsilon_0} \leq \varepsilon + 11\sqrt{\max(\rho_0(x, y), \rho_0(y, x))}.$$

The definition of ρ_0 gives unital completely positive maps

$$S_0 : \varphi_x(A(x)) \to \mathcal{O}_2 \quad \text{and} \quad T_0 : \varphi_y(A(y)) \to \mathcal{O}_2$$

such that

$$\|S_0(\varphi_x(u_l(x))) - \varphi_y(u_l(y))\| < \rho_0(x, y) + \varepsilon_0 \quad \text{and} \quad \|T_0(\varphi_y(u_l(y))) - \varphi_x(u_l(x))\| < \rho_0(y, x) + \varepsilon_0$$

for $1 \leq l \leq m$. Since $\mathcal{O}_2$ is nuclear, it follows from Proposition 0.4 that there are unital completely positive maps

$$S : \mathcal{O}_2 \to \mathcal{O}_2 \quad \text{and} \quad T : \mathcal{O}_2 \to \mathcal{O}_2$$

such that

$$\|S(\varphi_x(u_l(x))) - \varphi_y(u_l(y))\| < \rho_0(x, y) + 2\varepsilon_0 \text{ and } \|T(\varphi_y(u_l(y))) - \varphi_x(u_l(x))\| < \rho_0(y, x) + 2\varepsilon_0.$$

Proposition 1.7 provides isometries s, $t \in \mathcal{O}_2$ such that

$$\|s^*\varphi_x(u_l(x))s - \varphi_y(u_l(y))\| < \rho_0(x, y) + 3\varepsilon_0 \text{ and } \|t^*\varphi_y(u_l(y))t - \varphi_x(u_l(x))\| < \rho_0(y, x) + 3\varepsilon_0.$$

Now Lemma 1.12 provides a unitary $z \in \mathcal{O}_2 \otimes \mathcal{O}_2$ such that

$$\|z(1 \otimes \varphi_x(u_l(x)))z^* - 1 \otimes \varphi_y(u_l(y))\| < 11\sqrt{\max(\rho_0(x, y), \rho_0(y, x)) + 3\varepsilon_0}.$$

Let $\mu : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism (Theorem 0.8). Then $a \mapsto \mu(1 \otimes a)$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$ (by Proposition 0.7), so there is a unitary $w \in \mathcal{O}_2$ such that

$$\|w\mu(1 \otimes \varphi_x(u_l(x)))w^* - \varphi_x(u_l(x))\| < \varepsilon_0 \quad \text{and} \quad \|w\mu(1 \otimes \varphi_y(u_l(y)))w^* - \varphi_y(u_l(y))\| < \varepsilon_0$$

for $1 \leq l \leq m$. Set $v = w\mu(z)w^*$. Then one checks that

$$\begin{aligned} \|v\varphi_x(u_l(x))v^* - \varphi_y(u_l(y))\| &< 2\varepsilon_0 + 11\sqrt{\max(\rho_0(x, y), \rho_0(y, x)) + 3\varepsilon_0} \\ &\leq \varepsilon + 11\sqrt{\max(\rho_0(x, y), \rho_0(y, x))}. \end{aligned}$$

∎

The following two definitions will simplify the notation and terminology in several lemmas in the next section.

Definition 4.14. Let X be a topological space, and let A and B be two continuous fields of C*-algebras over X with given continuous sections $a_1, \dots, a_m$ of A and $b_1, \dots, b_m$ of B. If φ and ψ are representations of A and B in a C*-algebra D, then we define the *sectional distance* $d_{\mathrm{S}}(\varphi, \psi)$ between φ and ψ (with respect to $a_1, \dots, a_m$ and $b_1, \dots, b_m$) to be

$$d_{\mathrm{S}}(\varphi, \psi) = \sup_{x \in X} \max_{1 \leq l \leq m} \|\varphi_x(a_l(x)) - \psi_x(b_l(x))\|.$$

We suppress the sections a_l and b_l in the notation, since they will always be clear from the context. We use the same notation for representations $\varphi^{(1)}$ and $\varphi^{(2)}$ of two different restrictions $A|_{X\times\{y_1\}}$ and $A|_{X\times\{y_2\}}$ of a single continuous field over $X\times Y$ with a single set of sections $a_1,\ldots,a_m$:

$$d_{\mathrm{S}}(\varphi^{(1)},\varphi^{(2)})=\sup_{x\in X}\max_{1\leq l\leq m}\|\varphi_x^{(1)}(a_l(x,y_1))-\varphi_x^{(2)}(a_l(x,y_2))\|.$$

Sometimes $\varphi^{(1)}$ and $\varphi^{(2)}$ will be restrictions $\varphi|_{X\times\{y_1\}}$ and $\varphi|_{X\times\{y_2\}}$ of the same representation φ, and the obvious notation will also be used in this case.

Definition 4.15. Let X and Y be compact metric spaces, with metric d_Y on Y. Let A be a unital continuous field of C*-algebras over $X\times Y$, with continuous unitary sections $u_1,\ldots,u_m$ such that for each $(x,y)\in X\times Y$ the elements $u_1(x,y),\ldots,u_m(x,y)$ generate $A(x,y)$ as a C*-algebra. Let $\rho:[0,\infty)\to[0,\infty)$ be a nondecreasing function with $\lim_{t\to 0}\rho(t)=\rho(0)=0$. We say that A is (X,ρ)-*embeddable* (with respect to the sections $u_1,\ldots,u_m$) in a unital C*-algebra D if:

(1) For every $y\in Y$ there is an injective unital representation of $A|_{X\times\{y\}}$ in D.

(2) If $\varphi^{(1)}$ and $\varphi^{(2)}$ are injective unital representations of $A|_{X\times\{y_1\}}$ and $A|_{X\times\{y_2\}}$ in D, then for $\varepsilon>0$ there exists a continuous unitary function $w:X\to D$ such that the representation $w\varphi^{(1)}w^*$, given by $a\mapsto w(x)\varphi_x^{(1)}(a)w(x)^*$ for $a\in A(x)$, satisfies

$$d_{\mathrm{S}}(w\varphi^{(1)}w^*,\varphi^{(2)})<\varepsilon+\rho(d_Y(y_1,y_2)).$$

Remark 4.16. Let Y be a compact metric space, and let A be a unital continuous field of C*-algebras over Y, with $\Gamma(A)$ separable and exact. We can apply the terminology of the previous definition to A by taking X to be a one point space, so that $Y=X\times Y$. Proposition 4.13 now asserts that if m continuous unitary sections are given which generate $A(y)$ for each y, then A is (X,ρ)-embeddable in $\mathcal{O}_2$ for a suitable ρ. If ρ_0 is as in the statement of the proposition, then

$$\rho(r)=\sup_{d_Y(y_1,y_2)\leq r}11\sqrt{\max(\rho_0(y_1,y_2),\rho_0(y_2,y_1))}$$

will work. (The relation $\lim_{t\to 0}\rho(t)=0$ follows from the continuity of ρ_0 and the compactness of Y.)

5. Continuous embedding of exact continuous fields

Let A be a continuous field of C*-algebras over a compact metric space X, with $\Gamma(A)$ separable, exact, and unital. In the previous section, we have shown that the fibers of A over nearby points of X have embeddings in $\mathcal{O}_2$ which are close in a suitable sense. In this section, we use the methods of Haagerup and Rørdam [HR] to make, over a sufficiently nice space X, a continuous selection of these embeddings, so as to obtain a continuous representation of A in $\mathcal{O}_2$. It is not clear how general X can be in our argument, but certainly any compact manifold or finite CW complex is covered. The methods of Blanchard [Bl] cover more general spaces, but our approach has the advantage of giving better information about how close the embeddings of nearby fibers really are. We illustrate this for $X=[0,1]$, by showing that if the function ρ_0 of Proposition 4.11 (which gives an abstract distance between fibers) is Lip^{α}, then there is a $\mathrm{Lip}^{\alpha/2}$ representation of A in $\mathcal{O}_2$. In the next section, we will apply our results to give a $\mathrm{Lip}^{1/2}$ representation of the field of rotation algebras in $\mathcal{O}_2$, which is as good as the representation of this field in $L(H)$ in [HR].

The first five lemmas of this section are essentially one dimensional, with a parameter space carried along. They enable us to treat the case $X=[0,1]^n$ by induction on n. Four of these lemmas are simply modifications of lemmas in [HR], the modifications being the parameter space, the need to make do with approximate commutativity in some places where [HR] has exact commutativity, and the need to handle more general distance estimates than $\mathrm{Lip}^{1/2}$.

The following definition is useful for describing our version of Lemma 5.1 of [HR].

Definition 5.1. Let X be a topological space, let E be a Banach space, and let $[\alpha,\beta]$ be an interval in $\mathbb{R}$. A function $\xi:X\times[\alpha,\beta]\to E$ will be called *smooth in the* $[\alpha,\beta]$ *direction* if for every n the n-th derivative

$\dfrac{d^n}{dt^n}\xi(x,t)$ exists for every $(x,t) \in X \times [\alpha, \beta]$, and is jointly continuous in x and t. The function ξ is *piecewise smooth in the* $[\alpha, \beta]$ *direction* if there is a partition $\alpha = t_0 < t_1 < \cdots < t_n = \beta$ such that $\xi|_{X \times [t_{j-1}, t_j]}$ is smooth in the $[t_{j-1}, t_j]$ direction for $1 \leq j \leq n$.

It might be more appropriate to allow the break points t_j in the definition of piecewise smoothness to depend continuously on $x \in X$, but the definition we give suffices for our purposes.

Lemma 5.2. Let A be a unital C*-algebra, and let $B \subset A$ be a unital subalgebra with $B \cong \mathcal{O}_2$. Let X be a topological space, and let $v : X \to U(B' \cap A)$ be a continuous function from X to the unitary group of $B' \cap A$. Then there is a function $u : X \times [0,1] \to U(B)$ which is smooth in the $[0,1]$ direction and such that for all $x \in X$ we have:
(1) $u(x,0) = 1$ and $u(x,1) = v(x)$.
(2) $\left\| \frac{d}{dt} u(x,t) \right\| \leq 9$ for all $t \in [0,1]$.
(3) $\|[u(x,t), a]\| \leq 4\|[v(x), a]\|$ for all $t \in [0,1]$ and $a \in B' \cap A$.
(4) $\left\| \left[\frac{d}{dt} u(x,t), a \right] \right\| \leq 9\|[v(x), a]\|$ for all $t \in [0,1]$ and $a \in B' \cap A$.
(5) $\left\| \frac{d}{dt} \left[u(x,t) a u(x,t)^* \right] \right\| \leq 45\|[v(x), a]\|$ for all $t \in [0,1]$ and $a \in B' \cap A$.

Proof: The proof of Lemma 5.1 of [HR] works essentially as is, using $B' \cap A$ in place of M and $B \cong \mathcal{O}_2$ in place of M', and just carrying along the extra parameter x. The homomorphisms used there become π, $\rho : M_3 \to B$ and $\tilde{\pi}$, $\tilde{\rho} : M_3 \otimes (B' \cap A) \to A$. We take w and h as there, and the element called v there becomes $z(x) = \operatorname{diag}(v(x)^*, 1, v(x))$. We define

$$u(x,t) = \tilde{\pi}\left(\exp(ith) z(x) \exp(-ith) z(x)^* \right) \cdot \tilde{\rho}\left(\exp(ith) z(x) \exp(-ith) z(x)^* \right).$$

The proofs of the estimates are exactly the same as in [HR]. ∎

Lemma 5.3. Let X be a compact Hausdorff space, let $v : X \to U(\mathcal{O}_2)$ be continuous, let $S \subset \mathcal{O}_2$ be compact, and let $\varepsilon > 0$. Then there is a function $u : X \times [0,1] \to U(\mathcal{O}_2)$ which is piecewise smooth in the $[0,1]$ direction, and such that for all $x \in X$ we have:
(1) $u(x,0) = 1$ and $u(x,1) = v(x)$.
(2) $\left\| \frac{d}{dt} u(x,t) \right\| \leq 9 + \varepsilon$ for all $t \in [0,1]$.
(3) $\|[u(x,t), a]\| \leq 4\|[v(x), a]\| + \varepsilon$ for all $t \in [0,1]$ and $a \in S$.
(4) $\left\| \left[\frac{d}{dt} u(x,t), a \right] \right\| \leq 9\|[v(x), a]\| + \varepsilon$ for all $t \in [0,1]$ and $a \in S$.
(5) $\left\| \frac{d}{dt} [u(x,t) a u(x,t)^*] \right\| \leq 45\|[v(x), a]\| + \varepsilon$ for all $t \in [0,1]$ and $a \in S$.

Proof: Let $R = \sup_{a \in S} \|a\|$. Choose $\varepsilon_1 > 0$ so small that

$$(10 + 8R)\varepsilon_1 + \varepsilon_1^2 < \varepsilon.$$

Choose $0 < \delta < 1$ so small that

$$9\left(\frac{1}{1-\delta} - 1 \right) < \varepsilon_1 \quad \text{and} \quad 9\left(\frac{1}{1-\delta} - 1 \right) \cdot 2R < \frac{\varepsilon_1}{2}.$$

Choose $0 < \varepsilon_0 < 1$ so small that the quantities

$$6\varepsilon_0, \quad 6R\varepsilon_0, \quad \frac{18}{1-\delta} \cdot 6\varepsilon_0, \quad \frac{2}{\delta} \arcsin\left(\frac{3\varepsilon_0}{2} \right), \quad \text{and} \quad \frac{4}{\delta} \arcsin\left(\frac{3\varepsilon_0}{2} \right) R$$

are all less than ε_1. Choose a finite set $F \subset S$ such that every element of S is within ε_0 of an element of F, and a finite subset $G \subset U(\mathcal{O}_2)$ such that every $v(x)$, for $x \in X$, is within ε_0 of an element of G.

Let $\varphi : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism (Theorem 0.8). Then by Proposition 0.7 the homomorphism $a \to \varphi(1 \otimes a)$ is approximately unitarily equivalent to $\mathrm{id}_{\mathcal{O}_2}$. Therefore there is a unitary $w \in \mathcal{O}_2$ such that $\|w\varphi(1 \otimes b)w^* - b\| < \varepsilon_0$ for $b \in F \cup G$. Replacing φ by $w\varphi(\cdot)w^*$, we may assume that $\|\varphi(1 \otimes b) - b\| < \varepsilon_0$ for $b \in F \cup G$. If now $a \in S$, then consideration of $b \in F$ with $\|a - b\| < \varepsilon_0$ shows that $\|\varphi(1 \otimes a) - a\| < 3\varepsilon_0$. Similarly, $\|\varphi(1 \otimes v(x)) - v(x)\| < 3\varepsilon_0$ for all $x \in X$.

Use the previous lemma, with $A = \mathcal{O}_2 \otimes \mathcal{O}_2$ and $B = \mathcal{O}_2 \otimes \mathbb{C}$, to choose a smooth (in the $[0,1]$ direction) function $u_0 : X \times [0,1] \to U(\mathcal{O}_2 \otimes \mathcal{O}_2)$ for the unitary $x \mapsto 1 \otimes v(x)$. Define $h(x) = -i \log(v(x)^* \varphi(1 \otimes v(x)))$, and note that $\|v(x)^* \varphi(1 \otimes v(x)) - 1\| < 3\varepsilon_0$, so that $\|h(x)\| < 2 \arcsin(\frac{3\varepsilon_0}{2})$. Then set $u_1(x,t) = \exp(ith(x))\varphi(1 \otimes v(x))$. Now define

$$u(x,t) = \begin{cases} \varphi\left(u_0\left(x, \frac{t}{1-\delta}\right)\right) & 0 \le t \le 1-\delta \\ u_1\left(x, \frac{1}{\delta}(t-1)+1\right) & 1-\delta \le t \le 1. \end{cases}$$

Clearly $(x,t) \mapsto u(x,t)$ is a continuous and piecewise smooth (in the sense of Definition 5.1) unitary path from 1 to $v(x)$.

For $0 \le t \le 1-\delta$, we now estimate the quantities in parts (2) through (4) of the conclusion. We have

$$\left\| \frac{d}{dt} u(x,t) \right\| = \frac{1}{1-\delta} \left\| \frac{d}{dt} u_0(x,t) \right\| \le \frac{9}{1-\delta} \le 9 + \varepsilon_1.$$

Next, for $a \in S$, we have

$$\begin{aligned} \|[u(x,t), a]\| &\le 2\|\varphi(1 \otimes a) - a\| + \left\| \varphi\left(\left[u_0\left(x, \frac{t}{1-\delta}\right), 1 \otimes a \right] \right) \right\| \\ &\le 6\varepsilon_0 + 4\|[v(x), a]\| \le \varepsilon_1 + 4\|[v(x), a]\| \end{aligned}$$

and

$$\begin{aligned} \left\| \left[\frac{d}{dt} u(x,t), a \right] \right\| &\le 2 \left\| \frac{d}{dt} u(x,t) \right\| \|\varphi(1 \otimes a) - a\| + \left\| \varphi\left(\left[\frac{d}{dt}\left(u_0\left(x, \frac{t}{1-\delta}\right)\right), 1 \otimes a \right] \right) \right\| \\ &\le 6\varepsilon_0 \cdot \frac{9}{1-\delta} + \frac{1}{1-\delta} \cdot 9\|[v(x), a]\| \\ &\le \frac{\varepsilon_1}{2} + 9\left(\frac{1}{1-\delta} - 1 \right) \cdot 2R + 9\|[v(x), a]\| \le \varepsilon_1 + 9\|[v(x), a]\|. \end{aligned}$$

Estimating these same quantities for $1-\delta \le t \le 1$ instead, we obtain

$$\left\| \frac{d}{dt} u(x,t) \right\| \le \frac{1}{\delta}\|h\| \le \frac{2}{\delta} \arcsin\left(\frac{3\varepsilon_0}{2} \right) < \varepsilon_1 \le 9 + \varepsilon_1,$$

$$\begin{aligned} \|[u(x,t), a]\| &\le 2R\|u(x,t) - v(x)\| + \|[v(x), a]\| \le 2R\|\varphi(1 \otimes v(x)) - v(x)\| + \|[v(x), a]\| \\ &\le 6R\varepsilon_0 + \|[v(x), a]\| \le \varepsilon_1 + 4\|[v(x), a]\|, \end{aligned}$$

and

$$\left\| \left[\frac{d}{dt} u(x,t), a \right] \right\| = \frac{1}{\delta}\|[ih(x), a]\| \le \frac{2}{\delta}\|h\|\|a\| \le \frac{4}{\delta} \arcsin\left(\frac{3\varepsilon_0}{2} \right) R \le \varepsilon_1 + 9\|[v(x), a]\|.$$

We thus obtain in both cases, for $a \in S$,

$$\left\| \frac{d}{dt} u(x,t) \right\| \le 9 + \varepsilon_1, \quad \|[u(x,t), a]\| \le 4\|[v(x), a]\| + \varepsilon_1, \quad \text{and} \quad \left\| \left[\frac{d}{dt} u(x,t), a \right] \right\| \le 9\|[v(x), a]\| + \varepsilon_1.$$

Estimates (2), (3), and (4) follow since $\varepsilon_1 < \varepsilon$. Moreover, following the reasoning at the end of the proof of Lemma 5.1 of [HR], we obtain for $a \in S$ (using also $\varepsilon_0 \le 1$)

$$\begin{aligned} &\left\| \frac{d}{dt} [u(x,t) a u(x,t)^*] \right\| \\ &\le \left\| \left[\frac{d}{dt} u(x,t), a \right] \right\| \|u(x,t)^*\| + \|[u(x,t), a]\| \left\| \frac{d}{dt} u(x,t)^* \right\| \\ &\le 9\|[v(x), a]\| + \varepsilon_1 + (4\|[v(x), a]\| + \varepsilon_1)(9 + \varepsilon_1) \le 45\|[v(x), a]\| + (10 + 4\|[v(x), a]\|)\varepsilon_1 + \varepsilon_1^2 \\ &\le 45\|[v(x), a]\| + (10 + 8R)\varepsilon_1 + \varepsilon_1^2 < 45\|[v(x), a]\| + \varepsilon, \end{aligned}$$

which proves (5). ∎

Lemma 5.4. Let X be a compact metric space. Let A and B be two continuous fields of C*-algebras over $X \times [0,1]$, and let α and β be representations of A and B in $\mathcal{O}_2$. Let $u_1, \ldots, u_m$ be continuous unitary sections of A, let $v_1, \ldots, v_m$ be continuous unitary sections of B, and suppose there is $r > 0$ such that for every $t \in [0,1]$ there is a continuous unitary function $z : X \to U(\mathcal{O}_2)$ satisfying $d_{\mathrm{S}}(z(\alpha|_{X \times \{t\}})z^*, \beta|_{X \times \{t\}}) < r$. Then there is a continuous unitary function $w : X \times [0,1] \to U(\mathcal{O}_2)$ such that $d_{\mathrm{S}}(w\alpha w^*, \beta) < 10r$. Moreover, given continuous functions c_0, $c_1 : X \to U(\mathcal{O}_2)$ such that $d_{\mathrm{S}}(c_i(\alpha|_{X \times \{i\}})c_i^*, \beta|_{X \times \{i\}}) < r$ for $i = 0$, 1, we may choose w to satisfy $w(x,i) = c_i(x)$.

Proof: Choose a partition $0 = t_0 < t_1 < \cdots < t_n = 1$ of $[0,1]$ such that

$$d_{\mathrm{S}}(\alpha|_{X \times \{t\}}, \alpha|_{X \times \{t_j\}}) < \frac{r}{21} \quad \text{and} \quad d_{\mathrm{S}}(\beta|_{X \times \{t\}}, \beta|_{X \times \{t_j\}}) < \frac{r}{21}$$

for $t \in [t_{j-1}, t_j]$. By hypothesis, there are continuous unitary functions $z_0, \ldots, z_n : X \to U(\mathcal{O}_2)$ such that $d_{\mathrm{S}}(z_j(\alpha|_{X \times \{t_j\}})z_j^*, \beta|_{X \times \{t_j\}}) < r$. If unitaries c_i are given, take $z_0 = c_0$ and $z_n = c_1$. Now estimate

$$\begin{aligned} \|[z_{j-1}(x)^* z_j(x), \alpha_{x,t_j}(u_l(x,t_j))]\| &\le d_{\mathrm{S}}((z_{j-1}^* z_j)(\alpha|_{X \times \{t_j\}})(z_{j-1}^* z_j)^*, \alpha|_{X \times \{t_j\}}) \\ &\le d_{\mathrm{S}}(z_j(\alpha|_{X \times \{t_j\}})z_j^*, \beta|_{X \times \{t_j\}}) + d_{\mathrm{S}}(\beta|_{X \times \{t_j\}}, \beta|_{X \times \{t_{j-1}\}}) \\ &\quad + d_{\mathrm{S}}(z_{j-1}^*(\beta|_{X \times \{t_{j-1}\}})z_{j-1}, \alpha|_{X \times \{t_{j-1}\}}) + d_{\mathrm{S}}(\alpha|_{X \times \{t_{j-1}\}}, \alpha|_{X \times \{t_j\}}) \\ &< 2r + \frac{2r}{21}. \end{aligned}$$

Combining this with the estimates at the beginning of the proof, we get

$$\|[z_{j-1}(x)^* z_j(x), \alpha_{x,t}(u_l(x,t))]\| < 2r + \frac{4r}{21}$$

for $t \in [t_{j-1}, t_j]$, $x \in X$, and $1 \le l \le m$.

By Lemma 5.3, there are continuous unitary functions $\tilde{z} : X \times [t_{j-1}, t_j] \to \mathcal{O}_2$ with

$$\tilde{z}(x, t_{j-1}) = 1 \quad \text{and} \quad \tilde{z}(x, t_j) = z_{j-1}(x)^* z_j(x),$$

such that (from part (3) of the conclusion)

$$\begin{aligned} d_{\mathrm{S}}((\tilde{z}|_{X \times \{t\}})(\alpha|_{X \times \{t\}})(\tilde{z}|_{X \times \{t\}})^*, \alpha|_{X \times \{t\}}) &= \sup_{x \in X} \|[\tilde{z}(x,t), \alpha_{x,t}(u_l(x,t))]\| \\ &< 4\left(2r + \frac{4r}{21}\right) + \frac{r}{21} = 8r + \frac{17r}{21} \end{aligned}$$

for $t \in [t_{j-1}, t_j]$ and $1 \le l \le m$. (We can reparametrize the interval because we don't use the estimates on the derivatives.) Now define $w(x,t) = z_{j-1}(x)\tilde{z}(x,t)$ for $t \in [t_{j-1}, t_j]$. Then w is continuous, and for each t,

$$\begin{aligned} & d_{\mathrm{S}}((w|_{X \times \{t\}})(\alpha|_{X \times \{t\}})(w|_{X \times \{t\}})^*, \beta|_{X \times \{t\}}) \\ &\le d_{\mathrm{S}}((\tilde{z}|_{X \times \{t\}})(\alpha|_{X \times \{t\}})(\tilde{z}|_{X \times \{t\}})^*, \alpha|_{X \times \{t\}}) + d_{\mathrm{S}}(\alpha|_{X \times \{t\}}, \alpha|_{X \times \{t_{j-1}\}}) \\ &\quad + d_{\mathrm{S}}(\beta|_{X \times \{t\}}, \beta|_{X \times \{t_{j-1}\}}) + d_{\mathrm{S}}(z_{j-1}(\alpha|_{X \times \{t_{j-1}\}})z_{j-1}^*, \beta|_{X \times \{t_{j-1}\}}) \\ &< \left(8r + \frac{17r}{21}\right) + \frac{2r}{21} + \frac{2r}{21} + r = 10r, \end{aligned}$$

as desired. Furthermore, $w(x,0) = z_0(x)$ and $w(x,1) = z_n(x)$ for all x. ∎

The next lemma is an analog of Lemma 5.2 of [HR]. There is one additional twist, namely the number n', which is necessary in the absence of a Lipschitz condition.

Lemma 5.5. Let X be a compact metric space, let A be a unital continuous field of C*-algebras over $X \times [0,1]$, and let $u_1, \ldots, u_m$ be continuous unitary sections of A such that for each $(x,t) \in X \times [0,1]$ the elements $u_1(x,t), \ldots, u_m(x,t)$ generate $A(x,t)$ as a C*-algebra. Assume that A is (X, ρ)-embeddable (Definition 4.15) in $\mathcal{O}_2$ for some ρ, using the usual metric on $[0,1]$. Let $\varphi^{(0)}$ and $\varphi^{(1)}$ be injective unital representations of $A|_{X \times \{t_0\}}$ and $A|_{X \times \{t_1\}}$ in $\mathcal{O}_2$, with

$$d_{\mathrm{S}}(\varphi^{(0)}|_{X \times \{t_0\}}, \varphi^{(1)}|_{X \times \{t_1\}}) < d_0$$

for some $d_0 > \rho(t_1 - t_0)$. Let $0 < n' \leq n$ be integers, and set $s_j = t_0 + j(t_1 - t_0)/n$. Then there are injective unital representations $\gamma^{(j)}$ of $A|_{X \times \{s_j\}}$ such that $\gamma^{(0)} = \varphi^{(0)}$, $\gamma^{(n)} = \varphi^{(1)}$, and

$$d_{\mathrm{S}}(\gamma^{(j-1)}, \gamma^{(j)}) < 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'} \cdot d_0 \quad \text{and} \quad d_{\mathrm{S}}(\gamma^{(j)}, \varphi^{(0)}) < 91n'\rho\left(\frac{t_1 - t_0}{n}\right) + 91d_0$$

for all j.

Proof: The proof is easy if $\rho\left(\frac{t_1-t_0}{n}\right) = 0$. So assume $\rho\left(\frac{t_1-t_0}{n}\right) > 0$. Choose $\varepsilon > 0$ such that

$$\varepsilon < \frac{1}{2n'+1} \min\left(\rho\left(\frac{t_1 - t_0}{n}\right), d_0 - \rho(t_1 - t_0)\right).$$

Choose n'' and integers $0 = j(0) < j(1) < \cdots < j(n'') = n$ such that $n' \leq j(r) - j(r-1) < 2n'$ for all r. We construct the $\gamma^{(j)}$ for $j(r-1) < j \leq j(r)$ in blocks, by induction on r.

We start with $r = 1$. Using both the existence of injective unital representations and the sectional distance estimates for them in the definition of (X, ρ)-embeddability, construct injective unital representations $\beta^{(j)}$ of $A|_{X \times \{s_j\}}$ for $0 \leq j \leq j(1)$ such that

$$\beta^{(0)} = \varphi^{(0)} \quad \text{and} \quad d_{\mathrm{S}}(\beta^{(j-1)}, \beta^{(j)}) < \varepsilon + \rho\left(\frac{t_1 - t_0}{n}\right).$$

(This is done by induction. Take $\beta^{(0)} = \varphi^{(0)}$. Choose any injective unital representation β of $A|_{X \times \{s_1\}}$, and find $z \colon X \to U(\mathcal{O}_2)$ such that $d_{\mathrm{S}}(\beta^{(0)}, z\beta z^*) < \varepsilon + \rho\left(\frac{t_1-t_0}{n}\right)$. Set $\beta^{(1)} = z\beta z^*$. Then choose an injective unital representation β of $A|_{X \times \{s_2\}}$, etc.)

For $0 \leq i, j \leq j(1)$ we then have

$$d_{\mathrm{S}}(\beta^{(i)}, \beta^{(j)}) < |i - j|\left(\varepsilon + \rho\left(\frac{t_1 - t_0}{n}\right)\right).$$

The approximate unitary equivalence part of the hypotheses implies that there is a continuous function $w_1 \colon X \to U(\mathcal{O}_2)$ such that

$$d_{\mathrm{S}}(w_1 \beta^{(j(1))} w_1^*, \beta^{(0)}) < \varepsilon + \rho\left(j(1) \cdot \frac{t_1 - t_0}{n}\right) \leq \varepsilon + \rho(t_1 - t_0).$$

Combining the last two inequalities, we obtain, for $0 \leq j \leq j(1)$,

$$\begin{aligned}\sup_{x \in X} \max_{1 \leq l \leq m} \|[w_1(x), \beta_x^{(j)}(u_l(x, s_j))]\| &\leq d_{\mathrm{S}}(\beta^{(j)}, \beta^{(j(1))}) + d_{\mathrm{S}}(\beta^{(j)}, \beta^{(0)}) + \varepsilon + \rho(t_1 - t_0) \\ &< 2n'\rho\left(\frac{t_1 - t_0}{n}\right) + \rho(t_1 - t_0).\end{aligned}$$

Now apply Lemma 5.3 to obtain a function $\tilde{w}_1 \colon X \times [0,1] \to U(\mathcal{O}_2)$ which is piecewise smooth in the $[0,1]$ direction and satisfies

$$\tilde{w}_1(x, 0) = 1, \quad \tilde{w}_1(x, 1) = w_1(x), \quad \text{and} \quad \left\| \frac{d}{dt}\left[\tilde{w}_1(x,t) a \tilde{w}_1(x,t)^*\right] \right\| \leq 45\|[w_1(x), a]\| + \varepsilon$$

for all x and t, and for a in the compact set

$$S = \{\beta^{(j)}(u_l(x, s_j)) : 1 \leq l \leq m,\ 0 \leq j \leq j(1),\ x \in X\}.$$

Define

$$\gamma_x^{(j)}(a) = \tilde{w}_1\left(x, \frac{j}{j(1)}\right) \cdot \beta_x^{(j)}(a) \cdot \tilde{w}_1\left(x, \frac{j}{j(1)}\right)^*$$

for $a \in A(x, s_j)$ and $0 \leq j \leq j(1)$. This immediately gives $\gamma^{(0)} = \beta^{(0)} = \varphi^{(0)}$ and

$$d_{\mathrm{S}}(\gamma^{(j(1))}, \varphi^{(0)}) < \varepsilon + \rho(t_1 - t_0).$$

Moreover, using the derivative estimate from the previous paragraph, as well as $n' \leq j(1) < 2n'$, we obtain, for $1 \leq l \leq m$, $1 \leq j \leq j(1)$, and $x \in X$,

$$\begin{aligned}
&\|\gamma_x^{(j)}(u_l(x, s_j)) - \gamma_x^{(j-1)}(u_l(x, s_{j-1}))\| \\
&\quad \leq \ \|\beta_x^{(j)}(u_l(x, s_j)) - \beta_x^{(j-1)}(u_l(x, s_{j-1}))\| + \int_{(j-1)/j(1)}^{j/j(1)} \left\| \frac{d}{dt}\left[\tilde{w}_1(x,t)\beta_x^{(j)}(u_l(x, s_j))\tilde{w}_1(x,t)^*\right] \right\| dt \\
&\quad < \ \varepsilon + \rho\left(\frac{t_1 - t_0}{n}\right) + \frac{1}{j(1)}\left[45\left(2n'\rho\left(\frac{t_1 - t_0}{n}\right) + \rho(t_1 - t_0)\right) + \varepsilon\right] \\
&\quad \leq \ 2\varepsilon + 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{45}{n'}\rho(t_1 - t_0).
\end{aligned}$$

Thus

$$d_{\mathrm{S}}(\gamma^{(j-1)}, \gamma^{(j)}) < 2\varepsilon + 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{45}{n'}\rho(t_1 - t_0) < 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'} \cdot d_0.$$

The induction step for $r < n''$ is essentially the same. We start with $\gamma^{(j(r-1))}$ satisfying

$$d_{\mathrm{S}}(\gamma^{(j(r-1))}, \varphi^{(0)}) < \varepsilon + \rho(t_1 - t_0).$$

Construct, as at the beginning of the initial step, injective unital representations $\beta^{(j)}$ of $A|_{X \times \{s_j\}}$ for $j(r-1) \leq j \leq j(r)$ such that

$$\beta^{(j(r-1))} = \gamma^{(j(r-1))} \quad \text{and} \quad d_{\mathrm{S}}(\beta^{(j-1)}, \beta^{(j)}) < \varepsilon + \rho\left(\frac{t_1 - t_0}{n}\right).$$

(This redefines $\beta^{(j(r-1))}$ from the previous step, but we are done with the earlier one.) We obtain the same estimate on $d_{\mathrm{S}}(\beta^{(i)}, \beta^{(j)})$ as before, and then the approximate unitary equivalence part of the hypotheses provides $w_r \colon X \to U(\mathcal{O}_2)$ such that

$$d_{\mathrm{S}}(w_r\beta^{(j(r))}w_r^*, \varphi^{(0)}) < \varepsilon + \rho\left(j(r) \cdot \frac{t_1 - t_0}{n}\right) \leq \varepsilon + \rho(t_1 - t_0).$$

It follows that

$$d_{\mathrm{S}}(w_r\beta^{(j(r))}w_r^*, \beta^{(j(r-1))}) < 2(\varepsilon + \rho(t_1 - t_0)).$$

The commutator estimate in the initial step becomes

$$\begin{aligned}
\sup_{x \in X} \max_{1 \leq l \leq m} \|[w_r(x), \beta_x^{(j)}(u_l(x, s_j))]\| \ &\leq \ d_{\mathrm{S}}(\beta^{(j)}, \beta^{(j(r))}) + d_{\mathrm{S}}(\beta^{(j)}, \beta^{(j(r-1))}) + 2(\varepsilon + \rho(t_1 - t_0)) \\
&< \ 2n'\rho\left(\frac{t_1 - t_0}{n}\right) + 2\rho(t_1 - t_0)
\end{aligned}$$

for $j(r-1) \leq j \leq j(r)$. Now apply Lemma 5.3 as before, obtaining $\tilde{w}_r \colon X \times [0,1] \to U(\mathcal{O}_2)$. (In the definition of S, now use $j(r-1) \leq j \leq j(r)$.)

Define

$$\gamma_x^{(j)}(a) = \tilde{w}_r\left(x, \frac{j - j(r-1)}{j(r) - j(r-1)}\right) \cdot \beta_x^{(j)}(a) \cdot \tilde{w}_r\left(x, \frac{j - j(r-1)}{j(r) - j(r-1)}\right)^*$$

for $a \in A(x, s_j)$ and $j(r-1) \leq j \leq j(r)$. This gives the same $\gamma^{(j(r-1))}$ that we already have, and also immediately gives

$$d_{\mathrm{S}}(\gamma^{(j(r))}, \varphi^{(0)}) < \varepsilon + \rho(t_1 - t_0).$$

Set $k = j(r) - j(r-1) \geq n'$. The analog of the second last estimate in the initial step is then:

$$\begin{aligned}
&\|\gamma_x^{(j)}(u_l(x, s_j)) - \gamma_x^{(j-1)}(u_l(x, s_{j-1}))\| \\
&\quad \leq \|\beta_x^{(j)}(u_l(x, s_j)) - \beta_x^{(j-1)}(u_l(x, s_{j-1}))\| \\
&\qquad + \int_{(j-1-j(r-1))/k}^{(j-j(r-1))/k} \left\| \frac{d}{dt} \left[\tilde{w}_r(x,t) \beta_x^{(j)}(u_l(x, s_j)) \tilde{w}_r(x,t)^* \right] \right\| dt \\
&\quad < \varepsilon + \rho\left(\frac{t_1 - t_0}{n}\right) + \frac{1}{k}\left[45\left(2n'\rho\left(\frac{t_1 - t_0}{n}\right) + 2\rho(t_1 - t_0)\right) + \varepsilon\right] \\
&\quad \leq 2\varepsilon + 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'}\rho(t_1 - t_0).
\end{aligned}$$

Thus again

$$d_{\mathrm{S}}(\gamma^{(j)}, \gamma^{(j-1)}) < 2\varepsilon + 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'}\rho(t_1 - t_0) < 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'} \cdot d_0.$$

For the final step (with $r = n''$), we do things slightly differently. Choose $\beta^{(j)}$ for $j(n''-1) \leq j \leq j(n'') = n$ as before, but then choose $w_{n''}$ to satisfy

$$d_{\mathrm{S}}(w_{n''}\beta^{(n)}w_{n''}^*, \varphi^{(1)}) < \varepsilon.$$

(We have used $\varphi^{(1)}$ in place of $\varphi^{(0)}$. Both $\beta^{(n)}$ and $\varphi^{(1)}$ are injective representations of $A|_{X \times \{t_1\}}$ in $\mathcal{O}_2$.) This now gives

$$\begin{aligned}
d_{\mathrm{S}}(w_{n''}\beta^{(n)}w_{n''}^*, \beta^{(j(n''-1))}) &\leq d_{\mathrm{S}}(w_{n''}\beta^{(n)}w_{n''}^*, \varphi^{(1)}) + d_{\mathrm{S}}(\varphi^{(1)}, \varphi^{(0)}) + d_{\mathrm{S}}(\varphi^{(0)}, \beta^{(j(n''-1))}) \\
&< 2\varepsilon + \rho(t_1 - t_0) + d_0,
\end{aligned}$$

so

$$\begin{aligned}
\sup_{x \in X} \max_{1 \leq l \leq m} \|[w_{n''}(x), \beta^{(j)}(u_l(x, s_j))]\| &\leq d_{\mathrm{S}}(\beta^{(j)}, \beta^{(n)}) + d_{\mathrm{S}}(\beta^{(j)}, \beta^{(j(n''-1))}) + 2\varepsilon + \rho(t_1 - t_0) + d_0 \\
&< 2n'\rho\left(\frac{t_1 - t_0}{n}\right) + \rho(t_1 - t_0) + d_0.
\end{aligned}$$

Choose $\tilde{w}_{n''}$ as in the induction step, and define $\gamma^{(j)}$ as there for $j(n''-1) \leq j < n$. Define $\gamma^{(n)} = \varphi^{(1)}$. Set $k = n - j(n''-1) \geq n'$. Then the estimate of $d_{\mathrm{S}}(\gamma^{(j)}, \gamma^{(j-1)})$ in the induction step becomes, for $j(n''-1) \leq j < n$,

$$\begin{aligned}
&d_{\mathrm{S}}(\gamma^{(j-1)}, \gamma^{(j)}) \\
&\quad < \varepsilon + \rho\left(\frac{t_1 - t_0}{n}\right) + \frac{1}{j(n'') - j(n''-1)}\left[45\left(2n'\rho\left(\frac{t_1 - t_0}{n}\right) + \rho(t_1 - t_0) + d_0\right) + \varepsilon\right] \\
&\quad \leq 2\varepsilon + 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{45}{n'} \cdot (\rho(t_1 - t_0) + d_0).
\end{aligned}$$

For $j = n$, the corresponding estimate requires one extra term, namely $d_{\mathrm{S}}(w_{n''}\beta^{(n)}w_{n''}^*, \varphi^{(1)}) < \varepsilon$, so

$$d_{\mathrm{S}}(\gamma^{(n)}, \gamma^{(n-1)}) < 3\varepsilon + 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{45}{n'} \cdot (\rho(t_1 - t_0) + d_0).$$

In either case, we still have

$$d_{\mathrm{S}}(\gamma^{(j-1)}, \gamma^{(j)}) < 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'} \cdot d_0.$$

Our inductive construction is now complete. We have

$$d_{\mathrm{S}}(\gamma^{(j-1)}, \gamma^{(j)}) < 91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'} \cdot d_0$$

for all j with $1 \leq j \leq n$, and also, since $\varepsilon + \rho(t_1 - t_0) < d_0$, we have $d_{\mathrm{S}}(\gamma^{(j(r))}, \varphi^{(0)}) < d_0$ for all r. Now let j be arbitrary. Choose r such that $|j - j(r)| \leq n' - 1$. Then

$$\begin{aligned} d_{\mathrm{S}}(\gamma^{(j)}, \varphi^{(0)}) &\leq d_{\mathrm{S}}(\gamma^{(j)}, \gamma^{(j(r))}) + d_{\mathrm{S}}(\gamma^{(j(r))}, \varphi^{(0)}) \\ &\leq (n'-1)\left[91\rho\left(\frac{t_1 - t_0}{n}\right) + \frac{90}{n'} \cdot d_0\right] + d_0 < 91n'\rho\left(\frac{t_1 - t_0}{n}\right) + 91d_0, \end{aligned}$$

as desired. ∎

The following lemma is essentially the induction step in the main part of the proof of the theorem of this section. It is the analog of Lemma 5.3 and Theorem 5.4 of [HR].

Lemma 5.6. Let X and Y be compact metric spaces, with metric d_Y on Y. Let A be a unital continuous field of C*-algebras over $X \times [0,1] \times Y$, and suppose that there are continuous unitary sections $u_1, \ldots, u_m$ such that for each $\xi \in X \times [0,1] \times Y$ the elements $u_1(\xi), \ldots, u_m(\xi)$ generate $A(\xi)$ as a C*-algebra. If A is (X, ρ)-embeddable in $\mathcal{O}_2$, with respect to the metric $d_{[0,1]\times Y}((t_1, y_1), (t_2, y_2)) = |t_2 - t_1| + d_Y(y_1, y_2)$, then A is $(X \times [0,1], 10\rho)$-embeddable in $\mathcal{O}_2$, with respect to the metric d_Y.

Proof: Assuming the existence part of $(X \times [0,1], 10\rho)$-embeddability, the approximate unitary equivalence part follows directly from Lemma 5.4. We therefore prove existence.

We can assume without loss of generality that $\rho(r) > 0$ for $r > 0$. Fix $y_0 \in Y$; we construct an injective representation of $A|_{X\times[0,1]\times\{y_0\}}$ in $\mathcal{O}_2$. Choose positive integers n_1, $n_2, \ldots$ such that $n_k \geq 360$ and the integers $N_k = n_1 n_2 \cdots n_k$ (with $N_0 = 1$) satisfy

$$\rho\left(\frac{1}{N_{k+1}}\right) \leq \frac{1}{2} \cdot \rho\left(\frac{1}{N_k}\right)$$

for all k. Set $n' = 360$. Define $d_0 = 91\rho(1)$, and inductively define

$$d_{k+1} = 91\rho\left(\frac{1}{N_{k+1}}\right) + \frac{90}{n'} \cdot d_k.$$

Note that, by induction, we have $\rho\left(\frac{1}{N_k}\right) \leq 2^{-k}\rho(1)$. We can now inductively estimate d_k. First, $d_0 \leq 2 \cdot 91 \cdot \rho(1)$. Furthermore, if $d_k \leq 2^{-k+1} \cdot 91 \cdot \rho(1)$, then

$$d_{k+1} = 91\rho\left(\frac{1}{N_{k+1}}\right) + \frac{1}{4} \cdot d_k \leq 2^{-k-1} \cdot 91 \cdot \rho(1) + \frac{1}{4} \cdot 2^{-k+1} \cdot 91 \cdot \rho(1) = 2^{-k} \cdot 91 \cdot \rho(1).$$

Therefore $d_k \leq 2^{-k+1} \cdot 91 \cdot \rho(1) \leq 2^{-k} \cdot 182 \cdot \rho(1)$ for all k.

We now construct injective unital representations $\varphi^{(t)}$ of $A|_{X\times\{t\}\times\{y_0\}}$ in $\mathcal{O}_2$ for t in the set

$$S = \{j/N_k : k \geq 0,\ 0 \leq j \leq N_k\}.$$

These representations are required to satisfy

$$d_{\mathrm{S}}(\varphi^{((j-1)/N_k)}, \varphi^{(j/N_k)}) < d_k$$

for $1 \leq j \leq N_k$ and

$$d_{\mathrm{S}}(\varphi^{(j/N_k + r/N_{k+1})}, \varphi^{(j/N_k)}) < d_k + n' d_{k+1}$$

for $0 \leq j \leq N_k - 1$ and $0 \leq r \leq n_{k+1}$.

The construction is by induction on k. By hypothesis, there are injective unital representations $\varphi^{(t)}$ of $A|_{X\times\{t\}\times\{y_0\}}$ in $\mathcal{O}_2$ for $t = 0$ and $t = 1$. Since $\rho(1) < d_0$, the approximate unitary equivalence part of the hypothesis of (X, ρ)-embeddability implies that there is a continuous function $v : X \to \mathcal{O}_2$ such that $d_{\mathrm{S}}(\varphi^{(0)}, v\varphi^{(1)}v^*) < d_0$. Replacing $\varphi^{(1)}$ by $v\varphi^{(1)}v^*$, we may assume that $d_{\mathrm{S}}(\varphi^{(0)}, \varphi^{(1)}) < d_0$.

For the induction step, assume that $\varphi^{(j/N_k)}$ has been constructed for $0 \leq j \leq N_k$. Apply the previous lemma to each interval $[j/N_k, (j+1)/N_k]$, and call the resulting representations $\varphi^{(j/N_k + r/N_{k+1})}$ for $0 \leq r \leq n_{k+1}$. The estimates in the conclusion of that lemma give

$$d_{\mathrm{S}}(\varphi^{((i-1)/N_{k+1})}, \varphi^{(i/N_{k+1})}) < 91\rho\left(\frac{1}{N_{k+1}}\right) + \frac{90}{n'} \cdot d_k = d_{k+1}$$

for $jn_{k+1} < i \leq (j+1)n_{k+1}$ and

$$d_{\mathrm{S}}(\varphi^{(i/N_{k+1})}, \varphi^{(jn_{k+1}/N_{k+1})}) < 91n'\rho\left(\frac{1}{N_{k+1}}\right) + 91d_k = d_k + n'd_{k+1}$$

for $jn_{k+1} \leq i \leq (j+1)n_{k+1}$, as desired. This completes the inductive construction.

Note now that for $0 \leq r \leq n_{k+1}$ we have

$$\begin{aligned} d_{\mathrm{S}}(\varphi^{(j/N_k + r/N_{k+1})}, \varphi^{((j+1)/N_k)}) &\leq d_{\mathrm{S}}(\varphi^{(j/N_k + r/N_{k+1})}, \varphi^{(j/N_k)}) + d_{\mathrm{S}}(\varphi^{(j/N_k)}, \varphi^{((j+1)/N_k)}) \\ &< 2d_k + n'd_{k+1}. \end{aligned}$$

Let $0 \leq j_1 < j_2 \leq N_{k+1}$, and let i_1 and i_2 be the smallest and largest integers respectively that satisfy

$$\frac{j_1}{N_{k+1}} \leq \frac{i_1}{N_k} \quad \text{and} \quad \frac{i_2}{N_k} \leq \frac{j_2}{N_{k+1}}.$$

If $i_1 < i_2$, then

$$\begin{aligned} & d_{\mathrm{S}}(\varphi^{(j_1/N_{k+1})}, \varphi^{(j_2/N_{k+1})}) \\ &\leq d_{\mathrm{S}}(\varphi^{(j_1/N_{k+1})}, \varphi^{(i_1/N_k)}) + d_{\mathrm{S}}(\varphi^{(i_1/N_k)}, \varphi^{(i_2/N_k)}) + d_{\mathrm{S}}(\varphi^{(i_2/N_k)}, \varphi^{(j_2/N_{k+1})}) \\ &< 3d_k + 2n'd_{k+1} + d_{\mathrm{S}}(\varphi^{(i_1/N_k)}, \varphi^{(i_2/N_k)}). \end{aligned}$$

Otherwise, we have $i_1 = i_2$ or $i_1 = i_2 + 1$. In either case,

$$d_{\mathrm{S}}(\varphi^{(j_1/N_{k+1})}, \varphi^{(j_2/N_{k+1})}) \leq d_{\mathrm{S}}(\varphi^{(j_1/N_{k+1})}, \varphi^{(i_2/N_k)}) + d_{\mathrm{S}}(\varphi^{(i_2/N_k)}, \varphi^{(j_2/N_{k+1})}) \leq 3d_k + 2n'd_{k+1}.$$

(We actually get $2d_k + 2n'd_{k+1}$ if it happens that $i_1 = i_2 + 1$.) The second estimate necessarily holds whenever

$$\frac{j_2}{N_{k+1}} - \frac{j_1}{N_{k+1}} < \frac{1}{N_k}.$$

An induction argument therefore shows that if $0 \leq t_1 \leq t_2 \leq 1$ are in S and satisfy $t_2 - t_1 < \frac{1}{N_k}$, then (using $d_k \leq 2^{-k} \cdot 182 \cdot \rho(1)$ and $n' = 360$)

$$d_{\mathrm{S}}(\varphi^{(t_1)}, \varphi^{(t_2)}) < \sum_{s=k}^{\infty}(3d_s + 2n'd_{s+1}) \leq (6 + 2n') \cdot 182 \cdot \rho(1) \sum_{s=k+1}^{\infty} 2^{-s} \leq 133,000\rho(1) \cdot 2^{-k}.$$

Consider now, for each fixed l, the function from S to $C(X, \mathcal{O}_2)$ which sends t to $x \mapsto \varphi_x^{(t)}(u_l(x, t, y_0))$. The estimate of the previous paragraph implies that this function is uniformly continuous, and therefore extends by continuity to a function defined on all of $[0, 1]$, whose value at t we denote by $x \mapsto w_l(x, t)$. We now want to extend the map $u_l(x_0, t_0, y_0) \mapsto w_l(x_0, t_0)$ to a homomorphism $\psi_{(x_0,t_0)} : A(x_0, t_0, y_0) \to \mathcal{O}_2$. Let p be a polynomial in $2m$ noncommuting variables, and suppose

$$p(u_1(x_0, t_0, y_0), u_1(x_0, t_0, y_0)^*, \ldots, u_m(x_0, t_0, y_0), u_m(x_0, t_0, y_0)^*) = 0.$$

Then

$$t \mapsto p(u_1(x_0, t, y_0), u_1(x_0, t, y_0)^*, \ldots, u_m(x_0, t, y_0), u_m(x_0, t, y_0)^*)$$

is a continuous section of $A|_{\{x_0\} \times [0,1] \times \{y_0\}}$ which vanishes at (x_0, t_0, y_0). Considering a sequence (t_k) in S which converges to t_0, and using the fact that $\psi_{(x_0,t)}$ is the restriction of a homomorphism for $t \in S$, we see that

$$\begin{aligned} & p(w_1(x_0, t_0), w_1(x_0, t_0)^*, \ldots, w_m(x_0, t_0), w_m(x_0, t_0)^*) \\ &= \lim_{k \to \infty} \psi_{(x_0,t_k)}(p(u_1(x_0, t_k, y_0), u_1(x_0, t_k, y_0)^*, \ldots, u_m(x_0, t_k, y_0), u_m(x_0, t_k, y_0)^*)) = 0. \end{aligned}$$

It follows that $u_l(x_0, t_0, y_0) \mapsto w_l(x_0, t_0)$ extends to a homomorphism from the $*$-subalgebra generated by $u_1(x_0, t_0, y_0), \ldots, u_m(x_0, t_0, y_0)$ to $\mathcal{O}_2$. A similar approximation argument shows that this homomorphism is a contraction. Therefore it extends to the C*-algebra generated by these elements, which is $A(x_0, t_0, y_0)$ by

hypothesis. One checks directly that for a polynomial $p_{x,t}$ in $2m$ noncommuting variables and with coefficients varying continuously with (x,t), the function

$$\begin{aligned}(x,t) \quad \mapsto \quad & \psi_{x,t}(p_{x,t}(u_1(x,t,y_0),u_1(x,t,y_0)^*,\ldots,u_m(x,t,y_0),u_m(x,t,y_0)^*)) \\ & = p_{x,t}(w_1(x,t),w_1(x,t)^*,\ldots,w_m(x,t),w_m(x,t)^*)\end{aligned}$$

is continuous. A partition of unity argument shows that sections of the form

$$(x,t) \mapsto p_{x,t}(u_1(x,t,y_0),u_1(x,t,y_0)^*,\ldots,u_m(x,t,y_0),u_m(x,t,y_0)^*)$$

are dense in the set of all continuous sections of $A|_{X\times[0,1]\times\{y_0\}}$, and a standard argument now shows that $(x,t) \mapsto \psi_{(x,t)}(a(x,t))$ is continuous for any continuous section a of $A|_{X\times[0,1]\times\{y_0\}}$. Therefore ψ is a representation of $A|_{X\times[0,1]\times\{y_0\}}$ in $\mathcal{O}_2$. The maps $\psi_{(x,t)}$ are injective for $t \in S$ by construction, so ψ is injective by Lemma 4.6.

This completes the proof of the existence part of $(X \times [0,1], 10\rho)$-embeddability. ∎

Theorem 5.7. Let $X \subset [0,1]^n$ be a compact subset, and suppose there is an open set $U \subset [0,1]^n$ which contains X and a continuous retraction $f: U \to X$ such that $f|_X = \mathrm{id}_X$. Let A be a continuous field of C*-algebras over X, and assume that the section algebra $\Gamma(A)$ is separable and exact. Then A has an injective representation in $\mathcal{O}_2$, which can be taken unital if A is unital. Moreover, if A is unital then any two injective unital representations $\varphi^{(1)}$ and $\varphi^{(2)}$ of A are approximately unitarily equivalent in the sense that, for any finite set F of sections and any $\varepsilon > 0$, there is a continuous unitary function $u: X \to \mathcal{O}_2$ such that $\|u(x)\varphi_x^{(1)}(a(x))u(x)^* - \varphi_x^{(2)}(a(x))\| < \varepsilon$ for all $a \in F$ and $x \in X$.

Proof: We do the existence part first.

Unitizing A as in Lemma 4.2, we reduce to the unital case. (Exactness of the section algebra is preserved by Lemma 4.8.) Theorem 8 of [OZ] shows that $K \otimes \Gamma(A)$ is singly generated. So $(K \otimes \Gamma(A))^\dagger$ has a finite generating set, which we may take to consist of selfadjoint sections $a_1, \ldots, a_l$ satisfying $\|a_l\| < \pi$.

Let B be the continuous field over $[0,1]^n$ obtained by first forming the unitized tensor product $(K \otimes A)^\dagger$ following Lemmas 4.1 and 4.2, then constructing $f^*((K \otimes A)^\dagger)$ over U following Lemma 4.3, and finally extending over $[0,1]^n$ with fiber $\mathbb{C}$ at points not in U as in Lemma 4.4. Note that $B|_X \cong (K \otimes A)^\dagger$.

We must show that $\Gamma(B)$ is exact. First, $\Gamma(K \otimes A) \cong K \otimes \Gamma(A)$ is exact by Proposition 7.1 (iii) of [Kr4]. Then $\Gamma((K \otimes A)^\dagger)$ is exact, by Lemma 4.8 again. We now show that $\Gamma(B)$ is exact by showing that the maps $B(x_0) \to \Gamma(B)/\{b \in \Gamma(B) : b(x_0) = 0\}$ are locally liftable (Condition (2) of Theorem 4.7). If $x_0 \notin U$, then $B(x_0) = \mathbb{C}$, and this map lifts to a homomorphism to $\Gamma(B)$. If $x_0 \in U$, let $E \subset B(x_0)$ be a finite dimensional operator system. Note that $B(x_0) = ((K \otimes A)^\dagger)(f(x_0))$, so there is (by (1) implies (2) of Theorem 4.7) a unital completely positive map $T_0: E \to \Gamma((K \otimes A)^\dagger)$ which lifts the map

$$E \longrightarrow \Gamma((K \otimes A)^\dagger)/\{a \in \Gamma((K \otimes A)^\dagger) : a(f(x_0)) = 0\}.$$

Choose any state ω on E, and choose a continuous function $h_0: [0,1]^n \to [0,1]$ such that h_0 vanishes outside U and $h_0(x_0) = 1$. Then define $T: E \to \Gamma(B)$ by

$$T(b)(x) = h_0(x)T_0(b)(f(x)) + (1 - h_0(x))\omega(b) \cdot 1_{B(x)}.$$

This map lifts $E \to \Gamma(B)/\{b \in \Gamma(B) : b(x_0) = 0\}$, and is readily checked to be unital and completely positive. This completes the proof that $\Gamma(B)$ is exact.

Choose a continuous function $h: [0,1]^n \to [0,1]$ such that $h(x) > 0$ exactly when $x \in U$. Then the functions $x \mapsto h(x)a_l(f(x))$ (taken to be 0 for $x \notin U$) are continuous sections of B. Together with 1, their values at x generate $B(x)$ for all x. Therefore (since $\|a_l\| < \pi$) the sections $u_l(x) = \exp(ih(x)a_l(f(x)))$, for $1 \leq l \leq m$, also generate each $B(x)$.

For $0 \leq k \leq n$ write $[0,1]^n$ as $Y_k \times Z_k$ with $Y_k = [0,1]^k$ and $Z_k = [0,1]^{n-k}$. From Remark 4.16 it follows that B is (Y_0, ρ)-embeddable in $\mathcal{O}_2$ for a suitable ρ, with respect to the metric $d(x,y) = \sum_{j=1}^n |y_j - x_j|$. Using induction and Lemma 5.6, we find that B is $(Y_k, 10^k\rho)$-embeddable in $\mathcal{O}_2$ for all k. In particular, taking $k = n$, we see that B has an injective unital representation in $\mathcal{O}_2$.

Restriction to X gives an injective unital representation φ of $(K \otimes A)^\dagger$ in $\mathcal{O}_2$. Let $e \in K$ be a rank one projection, and define a continuous section p of $(K \otimes A)^\dagger$ by $p(x) = e \otimes 1_{A(x)}$. Then the function $x \mapsto \varphi_x(p(x))$ is a projection in $C(X, \mathcal{O}_2)$. It follows from [Zh2] that the set P of projections in $\mathcal{O}_2$ other than 0 and 1 is weakly

contractible (all homotopy groups trivial). Since it is homotopy equivalent to an open subset of a Banach space, it is contractible. Therefore the projection $x \mapsto \varphi_x(p(x))$ is homotopic to a constant projection $x \mapsto p_0$ for some $p_0 \in \mathcal{O}_2$. It follows that there is a unitary $v \in C(X, \mathcal{O}_2)$ such that $v(x)\varphi_x(p(x))v(x)^* = p_0$ for all $x \in X$. Then $a \mapsto v(x)\varphi_x(e \otimes a)v(x)^*$ is an injective unital homomorphism from $A(x)$ to $p_0\mathcal{O}_2p_0$, and the family of all these homomorphisms is an injective unital representation of A in $p_0\mathcal{O}_2p_0$. Since $p_0\mathcal{O}_2p_0 \cong \mathcal{O}_2$, the existence part is proved.

Now we do the approximate uniqueness part. Let the notation be as in the existence part. Choose a homomorphism $\mu_0 : K \otimes \mathcal{O}_2 \to \mathcal{O}_2$ which is an isomorphism onto a (nonunital) hereditary subalgebra of $\mathcal{O}_2$, and let $\mu : (K \otimes \mathcal{O}_2)^\dagger \to \mathcal{O}_2$ be the unital extension. Then the definition $\tilde{\varphi}^{(i)} = \mu \circ (\mathrm{id}_K \otimes \varphi_x^{(i)})$ gives two injective unital representations of $(K \otimes A)^\dagger$. Extend these to unital injective representations of B by setting $\psi_x^{(i)} = \tilde{\varphi}_{f(x)}^{(i)}$ for $x \in U$ and $\psi_x^{(i)}(\lambda) = \lambda \cdot 1$ for $x \notin U$. In the existence part of the proof, we saw that B is $(Y_n, 10^n\rho)$-embeddable in $\mathcal{O}_2$. It follows that $\psi^{(1)}$ is approximately unitarily equivalent to $\psi^{(2)}$. Restricting to X, we see that $\tilde{\varphi}^{(1)}$ is approximately unitarily equivalent to $\tilde{\varphi}^{(2)}$.

Let $F \in \Gamma(A)$ be finite, and assume all elements of F have norm at most 1. Regard F as a subset of $\Gamma((K \otimes A))^\dagger$ via $a \to p \otimes a$ (with p as above). Choose $\delta > 0$ such that $\delta < \frac{\varepsilon}{3}$, and also small enough that if projections q_1 and q_2 satisfy $\|q_1 - q_2\| < \delta$, then there is a unitary z with $zq_1z^* = q_2$ and $\|z - 1\| < \frac{\varepsilon}{3}$. Apply the definition of approximate unitary equivalence to $G = F \cup \{p\}$, using δ for ε. Call the resulting unitary w. Find z as above with $q_1 = wpw^*$ and $q_2 = p$. Then $u = pzwp$ is a unitary function with values in $\mathcal{O}_2 = \mathbb{C}e \otimes \mathcal{O}_2 \subset (K \otimes \mathcal{O}_2)^\dagger$ which satisfies $\|u(x)\varphi_x^{(1)}(a(x))u(x)^* - \varphi_x^{(2)}(a(x))\| < \varepsilon$ for all $a \in F$ and $x \in X$. ∎

We now want to extend this theorem to finite CW complexes. For this, we need a relative version. The following lemma is easy to prove and suffices.

Lemma 5.8. Let D^n be the closed unit ball in $\mathbb{R}^n$, with boundary S^{n-1}. Let A be a continuous field of C*-algebras over D^n, such that the section algebra $\Gamma(A)$ is separable and exact.

(1) Let φ be an injective representation of $A|_{S^{n-1}}$ in $\mathcal{O}_2$. Then there exists an injective representation ψ of A in $\mathcal{O}_2$ such that $\psi|_{S^{n-1}} = \varphi$. If A and φ are unital, so is ψ.

(2) Assume A is unital, and let $\varphi^{(1)}$ and $\varphi^{(2)}$ be injective unital representations of A in $\mathcal{O}_2$. Let $u_1, \dots, u_n$ be unitary sections of A, and let $z_0 : S^{n-1} \to U(\mathcal{O}_2)$ be continuous and satisfy $d_{\mathrm{S}}(z_0(\varphi^{(1)}|_{S^{n-1}})z_0^*, \varphi^{(2)}|_{S^{n-1}}) < \delta$. Then there is a continuous unitary $z : D^n \to U(\mathcal{O}_2)$ such that $z|_{S^{n-1}} = z_0$ and $d_{\mathrm{S}}(z\varphi^{(1)}z^*, \varphi^{(2)}) < 10\delta$.

Proof: It is convenient to define

$$S(r) = \{x \in D^n : \|x\| = r\} \quad \text{and} \quad S(r_1, r_2) = \{x \in D^n : r_1 \leq \|x\| \leq r_2\}.$$

We identify $S(r)$ with S^{n-1} in the obvious way. We further identify $S(r_1, r_2)$ with $S^{n-1} \times [0, 1]$ by starting at the outside edge: $S(r_2)$ goes to $S^{n-1} \times \{0\}$.

We first prove (1). Unitizing A (and correspondingly φ) as in the proof of the previous theorem, we may assume A and φ are unital. Let $\psi^{(0)}$ be an arbitrary injective unital representation of A in $\mathcal{O}_2$ (from the previous theorem). Choose a sequence $u_1, u_2, \dots$ of unitary sections of A such that $u_1(x), u_2(x), \dots$ generate $A(x)$ for every $x \in D^n$. Write $d_{\mathrm{S}}^{(n)}$ for the sectional distance with respect to $u_1, \dots, u_n$. Choose $0 < r_1 < r_2 < \cdots < 1$, with $r_n \to 1$, such that $d_{\mathrm{S}}^{(n)}(\psi^{(0)}|_{S(r)}, \psi^{(0)}|_{S(1)}) \leq 2^{-n}$ for $r_n \leq r \leq 1$. By the previous theorem, $\psi^{(0)}|_{S(1)}$ is approximately unitarily equivalent to φ, so there are continuous functions $w_{n,r} : S(r) \to U(\mathcal{O}_2)$, for $r_n \leq r \leq r_{n+1}$, such that $d_{\mathrm{S}}^{(n)}(w_{n,r}(\psi^{(0)}|_{S(r)})w_{n,r}^*, \varphi) \leq 2 \cdot 2^{-n}$. We may take $w_{1,r_1} = 1$. By Lemma 5.4, there exist $z_n : S(r_n, r_{n+1}) \to U(\mathcal{O}_2)$ such that

$$z_n|_{S(r_n)} = w_{n,r_n}, \quad z_n|_{S(r_{n+1})} = w_{n+1,r_{n+1}}, \quad \text{and} \quad d_{\mathrm{S}}^{(n)}(z_n(x)\psi_x^{(0)}z_n(x)^*, \varphi_{x/\|x\|}) \leq 20 \cdot 2^{-n}.$$

Define a continuous function z from the interior of D^n to $U(\mathcal{O}_2)$ by setting $z = z_n$ on $S(r_n, r_{n+1})$ and $z(x) = 1$ for $\|x\| < r_1$. Then set $\psi_x = \varphi_x$ for $\|x\| = 1$ and $\psi_x = z(x)\psi_x^{(0)}z(x)^*$ for $\|x\| < 1$. This defines the required ψ.

Now we prove (2). The previous theorem provides a continuous $w : D^n \to U(\mathcal{O}_2)$ such that $d_{\mathrm{S}}(w\varphi^{(1)}w^*, \varphi^{(2)}) < \delta$. Use Lemma 5.4 to find $z : S(\frac{1}{2}, 1) \to U(\mathcal{O}_2)$ such that

$$z|_{S^{n-1}} = z_0, \quad z|_{S(1/2)} = w|_{S(1/2)}, \quad \text{and} \quad d_{\mathrm{S}}(z(\varphi^{(1)}|_{S(1/2,1)})z^*, \varphi^{(2)}|_{S(1/2,1)})) < 10\delta.$$

Then define $z(x) = w(x)$ for $\|x\| < \frac{1}{2}$. ∎

Theorem 5.9. Let X be a finite CW complex. Let A be a continuous field of C*-algebras over X, such that $\Gamma(A)$ is separable and exact. Then A has an injective representation in $\mathcal{O}_2$, which can be taken unital if A is unital. Moreover, if A is unital then any two injective unital representations $\varphi^{(1)}$ and $\varphi^{(2)}$ of A are approximately unitarily equivalent in the sense of Theorem 5.7.

Proof: Both parts are proved by induction over the cells, and both are immediate from previous theorems for a zero dimensional finite CW complex.

For the existence induction step, assume the theorem is known for X, and let $Y = X \cup_f D_n$, where $f : S^{n-1} \to X$ is the attaching map. Let $g : D_n \to Y$ be the map extending f. Let φ be a (unital) injective representation of $A|_X$ in $\mathcal{O}_2$. Then $x \mapsto \varphi_{f(x)}$ is a (unital) injective representation of $g^*(A)|_{S^{n-1}}$ in $\mathcal{O}_2$. Now $g^*(A)$ is exact (one checks condition (2) of Theorem 4.7; it is easier than in the proof of Theorem 5.7) and has separable section algebra. The previous lemma therefore provides a (unital) injective representation of $g^*(A)$ in $\mathcal{O}_2$ which extends $x \mapsto \varphi_{f(x)}$. Use it to extend φ to a representation of A.

For the approximate uniqueness result, let a finite set F of sections and $\varepsilon > 0$ be given. Without loss of generality the sections in F are all unitary. Let N be the number of cells of strictly positive dimension, and let X_0 be the zero skeleton. Given $\varphi^{(1)}$ and $\varphi^{(2)}$, choose a unitary $v_0 : X_0 \to \mathcal{O}_2$ such that $d_{\mathrm{S}}(v_0(\varphi^{(1)}|_{X_0})v_0^*, \varphi^{(2)}|_{X_0}) < 10^{-N}\varepsilon$. Use the uniqueness part of the previous lemma, in the same way the existence part was used in the previous paragraph, to extend v_0 cell by cell. If X_n is the subcomplex obtained by adding n cells and v_n is the unitary defined on it, we will have $d_{\mathrm{S}}(v_n(\varphi^{(1)}|_{X_n})v_n^*, \varphi^{(2)}|_{X_n}) < 10^{-N+n}\varepsilon$. ∎

The advantage of the methods of this section is that they give control over the "smoothness" of the images under the representation of the generating sections. Here is the one dimensional version, which is easy to prove.

Theorem 5.10. Let A be a unital continuous field over $[0,1]$ such that $\Gamma(A)$ is exact, and let $u_1, \ldots, u_m \in \Gamma(A)$ be unitary sections of A such that, for each $x \in X$, the elements $u_1(x), \ldots, u_m(x)$ generate $A(x)$. Suppose the function ρ_0 of Proposition 4.11 (using $u_1, \ldots, u_m$ in place of $a_1, \ldots, a_m$) is Lip^α for some $\alpha \in (0,1]$, that is, there is a constant C_0 such that $\rho_0(x,y) \le C_0|x-y|^\alpha$ for all $x,\ y \in [0,1]$. Then there is a unital injective representation φ of A in $\mathcal{O}_2$ such that the the functions $x \mapsto \varphi_x(u_l(x))$ are $\mathrm{Lip}^{\alpha/2}$, that is, there is a constant C such that $\|\varphi_x(u_l(x)) - \varphi_y(u_l(y))\| \le C|x-y|^{\alpha/2}$ for all $x,\ y \in [0,1]$. Moreover, C depends only on C_0 and α.

Proof: By Proposition 4.13, the function ρ defined there satisfies

$$\rho(x,y) \le 11C_0^{1/2}|x-y|^{\alpha/2}.$$

That is, taking X_0 to be a one point space, A is (X_0, ρ)-embeddable in $\mathcal{O}_2$ with $\rho(t) = 11C_0^{1/2}t^{\alpha/2}$. (See Definition 4.15 and Remark 4.16.)

We now follow the proof of Lemma 5.6, with the spaces X and Y there both one point spaces, and making suitable minor modifications. Set $C_1 = 11C_0^{1/2}$, and set

$$\beta = \frac{1}{1-\alpha/2} \in [1,2].$$

Choose some integer n with $181^\beta \le n \le 181^\beta + 1$. Choose the numbers of the proof of Lemma 5.6 to be $n_1 = n_2 = \cdots = n' = n$, so that $N_k = n^k$. Take $d_0 = 181C_1$, and define $d_{k+1} = 91\rho(n^{-(k+1)}) + 90n^{-1}d_k$ inductively, as in the proof of Lemma 5.6. We prove by induction that $d_k \le 181C_1 n^{-k\alpha/2}$. The one slightly nontrivial step is the observation that

$$181n^{-(1-\alpha/2)} \le 181^{1-\beta(1-\alpha/2)} = 1.$$

Following the procedure of the proof of Lemma 5.6, we now obtain, for $n^{-(k+1)} < t_2 - t_1 \le n^{-k}$, the estimate

$$\begin{aligned} d_{\mathrm{S}}(\varphi^{(t_1)}, \varphi^{(t_2)}) &< \sum_{s=k}^{\infty}(3d_s + 2nd_{s+1}) \le 3 \cdot 181C_1 \cdot \frac{n^{-k\alpha/2}}{1-n^{-\alpha/2}} + 2n \cdot 181C_1 \cdot \frac{n^{-(k+1)\alpha/2}}{1-n^{-\alpha/2}} \\ &= \frac{181C_1(3n^{\alpha/2}+2n)}{1-n^{-\alpha/2}} \cdot n^{-(k+1)\alpha/2} \le \frac{181C_1\left(5(181^\beta+1)\right)}{1-181^{-\alpha\beta/2}} \cdot (t_2-t_1)^{\alpha/2}. \end{aligned}$$

The rest of the proof goes through as it stands, and we obtain in the end

$$\|\psi_{t_1}(u_l(t_1)) - \psi_{t_2}(u_l(t_2))\| \leq M(\alpha) C_0^{1/2} |t_1 - t_2|^{\alpha/2},$$

with

$$M(\alpha) = 11 \cdot \frac{181 \cdot 5(181^{\beta} + 1)}{1 - 181^{-\alpha\beta/2}}.$$

This proves the theorem with $C = M(\alpha) C_0^{1/2}$. ∎

Remark 5.11. With more care, the choice of $M(\alpha)$ in the proof of Theorem 5.10 can be improved considerably. To keep down the sizes of some of the numbers in the next section, we describe how to get $C \leq 330{,}000 C_0^{1/2}$ in the case $\alpha = 1$. First, taking $n' = n$ in the proof of Lemma 5.5 allows considerable simplification (this is the case done in [HR]) and improvement of the conclusion to

$$d_{\mathrm{S}}(\gamma^{(j-1)}, \gamma^{(j)}) < 46 \cdot \rho\left(\frac{t_1 - t_0}{n}\right) + \frac{45}{n} \cdot d_0.$$

Let C_1 be as in the proof of Theorem 5.10, but take $n = n' = 90^2$ (as in [HR]). Take $d_0 = 46C_1$ and $d_{k+1} = 46\rho(n^{-(k+1)}) + 45n^{-1}d_k$. Then $d_k \leq 92C_1 n^{-k/2}$. For t_1 and t_2 of the form j_1/n^{k_1} and j_2/n^{k_2}, estimate $d_{\mathrm{S}}(\varphi^{(t_1)}, \varphi^{(t_2)})$ by the more careful method in the proof of Theorem 5.4 of [HR]. One obtains

$$d_{\mathrm{S}}(\varphi^{(t_1)}, \varphi^{(t_2)}) \leq 320 \cdot 92 \cdot C_1 \cdot |t_2 - t_1|^{1/2} \leq 30{,}000 C_1 |t_2 - t_1|^{1/2}.$$

6. The field of rotation algebras

In this section, we apply the results of the previous section specifically to the continuous field of rotation algebras. In Theorem 5.4 and Corollary 5.5 of [HR], Haagerup and Rørdam construct a $\mathrm{Lip}^{1/2}$ (with respect to the sections given by the standard generators) representation of the field of rotation algebras in $L(H)$ for a separable infinite dimensional Hilbert space H. In this section, we produce a $\mathrm{Lip}^{1/2}$ representation in $\mathcal{O}_2$.

By combining the estimates of Haagerup and Rørdam (Theorem 4.9 (1) of [HR]), the explicit estimate in Lemma 1.8, an easy computation to show that the constant M there is 1, and Lemma 1.10, one finds that the function ρ_0 of Proposition 4.11 satisfies $\rho_0(\theta_1, \theta_2) \leq 480 \cdot |\theta_1 - \theta_2|^{1/2}$. A slight modification of Theorem 5.10 (to use the circle instead of $[0, 1]$) then gives a $\mathrm{Lip}^{1/4}$ representation in $\mathcal{O}_2$.

We improve this procedure by estimating $\rho_0(\theta_1, \theta_2)$ directly. By explicitly estimating the completely bounded norms of certain linear maps between finite dimensional operator spaces in the rotation algebras, we prove an inequality of the form $\rho_0(\theta_1, \theta_2) \leq C_0|\theta_1 - \theta_2|$ (not just $C_0|\theta_1 - \theta_2|^{1/2}$) for some constant C_0. This implies the existence of a $\mathrm{Lip}^{1/2}$ representation in $\mathcal{O}_2$. We therefore have an alternate proof (with different constants) of Theorems 4.9 and 5.4 of [HR]. This proof makes no use of unbounded operators or canonical commutation relations.

We begin by establishing notation.

Notation 6.1. For $\theta \in \mathbb{R}$ let $A(\theta)$ be the rotation C*-algebra, the universal C*-algebra generated by unitaries $u(\theta)$ and $v(\theta)$ satisfying $u(\theta)v(\theta) = \exp(2\pi i\theta)v(\theta)u(\theta)$. By Corollary 3.6 of [Rf], the rotation algebras are the fibers of a continuous field over the circle S^1, which we think of as $[0, 1]$ with the endpoints identified, or as $\mathbb{R}/\mathbb{Z}$. Moreover, u and v are continuous sections. (The fact that the rotation algebras form a continuous field in this manner was known to Elliott and others before [Rf].) Let $E(\theta) \subset A(\theta)$ be the operator space $E(\theta) = \mathrm{span}(1, u(\theta), u(\theta)^*, v(\theta), v(\theta)^*)$.

Further, for each $\theta \in \mathbb{R}$, fix a unital embedding $\iota_\theta : A(\theta) \to \mathcal{O}_2$, and use it to regard $A(\theta)$ as a unital subalgebra of $\mathcal{O}_2$. Define

$$\rho_0(\theta_1, \theta_2) = \inf_T (\max(\|T(u(\theta_1)) - \iota_{\theta_2}(u(\theta_2))\|, \|T(v(\theta_1)) - \iota_{\theta_2}(v(\theta_2))\|)),$$

where the infemum is taken over all unital completely positive maps $T : A(\theta_1) \to \mathcal{O}_2$. By Proposition 4.11 (1), this function does not depend on the embeddings used. (In the case at hand, since all algebras involved are nuclear, Remark 4.12 implies that one gets the same function by taking the infemum over all unital completely positive maps $T : A(\theta_1) \to A(\theta_2)$.)

The following two lemmas constitute an analog of Proposition 4.5 of [HR].

Lemma 6.2. The function ρ_0 is continuous and translation invariant, that is, $\rho_0(\theta_1+\theta, \theta_2+\theta) = \rho_0(\theta_1, \theta_2)$ for all θ_1, θ_2, $\theta \in \mathbb{R}$.

Proof: The function ρ_0 is continuous by Proposition 4.11 (2). For translation invariance, it suffices to prove that $\rho_0(\theta_1+\theta, \theta_2+\theta) \leq \rho_0(\theta_1, \theta_2)$. By continuity, we may restrict to θ_1, θ_2 irrational and θ rational.

Let $\varepsilon > 0$, and let $T : A(\theta_1) \to \mathcal{O}_2$ satisfy

$$\max(\|T(u(\theta_1)) - \iota_{\theta_2}(u(\theta_2))\|, \|T(v(\theta_1)) - \iota_{\theta_2}(v(\theta_2))\|) < \rho_0(\theta_1, \theta_2) + \frac{\varepsilon}{2}.$$

Set $u_1 = u(\theta_1) \otimes u(\theta)$ and $v_1 = v(\theta_1) \otimes v(\theta)$, which are unitaries in $A(\theta_1) \otimes A(\theta)$ satisfying $u_1 v_1 = \exp(2\pi i(\theta_1 + \theta)) v_1 u_1$. Since $\theta_1 + \theta$ is irrational, this gives a unital embedding λ of $A(\theta_1+\theta)$ in $A(\theta_1) \otimes A(\theta)$. Similarly, the unitaries $u_2 = u(\theta_2) \otimes u(\theta)$ and $v_2 = v(\theta_2) \otimes v(\theta)$ give a unital embedding ψ of $A(\theta_2+\theta)$ in $A(\theta_2) \otimes A(\theta)$. We will estimate $\rho_0(\theta_1+\theta, \theta_2+\theta)$ by considering the unital completely positive map $T \otimes \mathrm{id}_{A(\theta)}$ followed by a suitable embedding in $\mathcal{O}_2$.

Let $\mu : \mathcal{O}_2 \otimes \mathcal{O}_2 \to \mathcal{O}_2$ be an isomorphism (from Theorem 0.8), and let

$$\varphi = \mu \circ (\iota_{\theta_2} \otimes \iota_\theta) \circ \psi : A(\theta_2 + \theta) \to \mathcal{O}_2.$$

Then φ is approximately unitarily equivalent to $\iota_{\theta_2+\theta}$ by Theorem 1.15, so there is a unitary $w \in \mathcal{O}_2$ such that $\|w\varphi(a)w^* - \iota_{\theta_2+\theta}(a)\| \leq \frac{1}{2}\varepsilon\|a\|$ for $a \in E(\theta_2+\theta)$.

Now define $S : A(\theta_1+\theta) \to \mathcal{O}_2$ by

$$S_0 = \mu \circ (T \otimes \iota_\theta) \circ \lambda \quad \text{and} \quad S(a) = w S_0(a) w^*.$$

This map is unital and completely positive because $T \otimes \mathrm{id}_{A(\theta)}$ is (by Proposition IV.4.23 (i) of [Tk]). Furthermore, it is easy to check that

$$\begin{aligned}
&\|S(u(\theta_1+\theta)) - \iota_{\theta_2+\theta}(u(\theta_2+\theta))\| \\
&\quad\leq \quad \|S_0(u(\theta_1+\theta)) - \varphi(u(\theta_2+\theta))\| + \|w\varphi(u(\theta_2+\theta))w^* - \iota_{\theta_2+\theta}(u(\theta_2+\theta))\| \\
&\quad< \quad \|T(u(\theta_1)) \otimes u(\theta) - \iota_{\theta_2}(u(\theta_2)) \otimes u(\theta)\| + \frac{\varepsilon}{2} < \rho_0(\theta_1, \theta_2) + \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \rho_0(\theta_1, \theta_2) + \varepsilon,
\end{aligned}$$

and similarly for v in place of u. This shows that $\rho_0(\theta_1+\theta, \theta_2+\theta) < \rho_0(\theta_1, \theta_2) + \varepsilon$. ∎

Lemma 6.3. The function ρ_0 is a continuous pseudometric on $\mathbb{R}$.

Proof: Continuity was shown in the previous lemma. The triangle inequality is Proposition 4.11 (3). It remains to prove symmetry. It suffices to prove that $\rho_0(0, -\theta) = \rho_0(0, \theta)$. Indeed, the previous lemma then gives

$$\rho_0(\theta_2, \theta_1) = \rho_0(0, \theta_1 - \theta_2) = \rho_0(0, \theta_2 - \theta_1) = \rho_0(\theta_1, \theta_2)$$

for any θ_1 and θ_2.

We may clearly assume $\theta \neq 0$. Let $\varphi_\theta : A(\theta) \to A(-\theta)$ be the isomorphism given by

$$\varphi_\theta(u(\theta)) = v(\theta) \quad \text{and} \quad \varphi_\theta(v(\theta)) = u(\theta).$$

Since $\rho(0, \theta)$ does not depend on the choice of embedding ι_θ of $A(\theta)$ in $\mathcal{O}_2$, we may assume that $\iota_{-\theta} = \iota_\theta \circ \varphi_\theta^{-1}$, that is, that in $\mathcal{O}_2$ we have

$$\iota_{-\theta}(u(-\theta)) = \iota_\theta(v(\theta)) \quad \text{and} \quad \iota_{-\theta}(v(-\theta)) = \iota_\theta(u(\theta)).$$

Given a unital completely positive map $T : A(0) \to \mathcal{O}_2$, the map $T \circ \varphi_0 : A(0) \to \mathcal{O}_2$ satisfies

$$\|(T \circ \varphi_0)(u(0)) - \iota_{-\theta}(u(-\theta))\| = \|T(v(0)) - \iota_\theta(v(\theta))\|$$

and

$$\|(T \circ \varphi_0)(v(0)) - \iota_{-\theta}(v(-\theta))\| = \|T(u(0)) - \iota_\theta(u(\theta))\|.$$

Taking the infemum over all T, we find that $\rho_0(0, -\theta) \leq \rho_0(0, \theta)$. Replacing θ by $-\theta$, we get equality. ∎

We now prepare to estimate $\rho_0(0, \theta)$.

Lemma 6.4. Let H be a Hilbert space, and let $\xi, \eta \in H$ with $\|\xi\| = \|\eta\| = 1$. Then $\mathrm{Re}(\langle \xi, \eta \rangle) = 1 - \frac{1}{2}\|\xi - \eta\|^2$.

Proof: Write $\|\xi - \eta\|^2 = \langle \xi - \eta, \xi - \eta \rangle$ and calculate. ∎

Lemma 6.5. Let $E_0(\theta) = \mathrm{span}(1, u(\theta), v(\theta)) \subset A(\theta)$. Define $T_\theta : E_0(\theta) \to E_0(0)$ by

$$T_\theta(1) = 1, \quad T_\theta(u(\theta)) = u(0), \quad \text{and} \quad T_\theta(v(\theta)) = v(0).$$

If θ is a rational number of the form $m^2/(2n+1)^2$ with m, $n \in \mathbb{N}$ (not necessarily in lowest terms), and if $\theta < 2/25$, then $\|T_\theta\|_{\mathrm{cb}} \leq (1 - \frac{25}{2}\theta)^{-1/2}$.

Proof: We start by making several reductions. First, note that if E is an operator space, then a bounded linear functional $\omega : E \to \mathbb{C}$ is completely bounded and satisfies $\|\omega\|_{\mathrm{cb}} = \|\omega\|$. (See, for example, Proposition 3.7 of [Pl].) Therefore, using $A(0) = C(S^1 \times S^1)$, we have

$$\|T_\theta\|_{\mathrm{cb}} = \sup_{x \in S^1 \times S^1} \| \mathrm{ev}_x \circ T_\theta \|_{\mathrm{cb}} = \sup_{x \in S^1 \times S^1} \| \mathrm{ev}_x \circ T_\theta \| = \|T_\theta\|.$$

It therefore suffices to show that $\|T_\theta\| \leq (1 - \frac{25}{2}\theta)^{-1/2}$.

For the next step, it is convenient to use the (nonstandard) convention $\mathrm{sgn}(0) = 1$. Our problem is equivalent to showing that for α, β, $\gamma \in \mathbb{C}$ we have:

$$\|\alpha \cdot 1 + \beta u(0) + \gamma v(0)\| = 1 \quad \text{implies} \quad \|\alpha \cdot 1 + \beta u(\theta) + \gamma v(\theta)\| \geq \left(1 - \tfrac{25}{2}\theta\right)^{1/2}.$$

Multiplying the two expressions inside the norm signs by $\overline{\mathrm{sgn}(\alpha)}$, we see that it suffices to prove this with $\alpha \geq 0$. Next, note that clearly $\|\alpha \cdot 1 + \beta u(0) + \gamma v(0)\| \leq \alpha + |\beta| + |\gamma|$, while the reverse inequality follows by considering the point $x = \left(\overline{\mathrm{sgn}(\beta)}, \overline{\mathrm{sgn}(\gamma)}\right) \in S^1 \times S^1$. So it suffices to show that for α, β, $\gamma \in \mathbb{C}$ with $\alpha \geq 0$ we have

$$\alpha + |\beta| + |\gamma| = 1 \quad \text{implies} \quad \|\alpha \cdot 1 + \beta u(\theta) + \gamma v(\theta)\| \geq \left(1 - \tfrac{25}{2}\theta\right)^{1/2}.$$

Now let $q = (2n+1)^2$, and define $\zeta = \exp(2\pi i/q)$, a primitive q-th root of 1. Define unitaries y_0, $z_0 \in M_q$ by

$$y_0 = \mathrm{diag}(1, \zeta, \ldots, \zeta^{q-1}) \quad \text{and} \quad z_0 = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix}.$$

Then $y_0 z_0 = \zeta z_0 y_0$. Set $y = y_0^m$ and $z = z_0^m$, so that $yz = \zeta^{m^2} zy = \exp(2\pi i\theta) zy$. Thus there is a unital homomorphism $\varphi : A(\theta) \to M_q$ given by $\varphi(u(\theta)) = \overline{\mathrm{sgn}(\beta)} y$ and $\varphi(v(\theta)) = \overline{\mathrm{sgn}(\gamma)} z$. We have

$$\varphi(\alpha \cdot 1 + \beta u(\theta) + \gamma v(\theta)) = \alpha \cdot 1 + |\beta| y + |\gamma| z.$$

Note that we can write $\theta = m^2/(2n+1)^2$ with n arbitrarily large. Since $\|\varphi\| \leq 1$, it therefore suffices to show that for α, β, $\gamma \in \mathbb{C}$ with $\alpha \geq 0$ and for all large enough n, we have

$$\alpha + |\beta| + |\gamma| = 1 \quad \text{implies} \quad \|\alpha \cdot 1 + |\beta| y + |\gamma| z\| \geq \left(1 - \tfrac{25}{2}\theta\right)^{1/2}.$$

Equivalently, we assume that α, β, $\gamma \geq 0$, and show that

$$\alpha + \beta + \gamma = 1 \quad \text{implies} \quad \|\alpha \cdot 1 + \beta y + \gamma z\| \geq \left(1 - \tfrac{25}{2}\theta\right)^{1/2}$$

for large n.

Define

$$\xi_0 = \left(1, \frac{n-1}{n}, \ldots, \frac{2}{n}, \frac{1}{n}, 0, 0, \ldots, 0, \frac{1}{n}, \frac{2}{n}, \ldots, \frac{n-1}{n}\right) \in \mathbb{C}^q \quad \text{and} \quad \xi = \frac{1}{\|\xi_0\|}\xi_0.$$

Using the formula

$$1^2 + 2^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1),$$

we can calculate

$$\|\xi_0\|^2 = 1 + 2\sum_{k=1}^{n-1} \left(\frac{k}{n}\right)^2 = 1 + \tfrac{1}{3}n^{-1}(n+1)(2n+1) \geq \tfrac{2}{3}n.$$

Further, using

$$|1-\zeta^{km}| \leq km|1-\zeta| \leq \frac{2\pi km}{(2n+1)^2}$$

and $(n-k)k \leq n^2/4$ for $0 \leq k \leq n$, we obtain

$$\begin{aligned} \|\xi_0 - y\xi_0\|^2 &= 2\sum_{k=1}^{n-1}\left(\frac{n-k}{n}\right)^2 |1-\zeta^{km}|^2 \\ &\leq \left(\frac{2}{n^2}\right)\left(\frac{4\pi^2 m^2}{(2n+1)^4}\right)\sum_{k=1}^{n-1}(n-k)^2k^2 \leq \left(\frac{\pi^2}{2}\right)\left(\frac{(n-1)n^2m^2}{(2n+1)^4}\right). \end{aligned}$$

So

$$\|\xi_0 - y\xi_0\| \leq \frac{\pi nm}{(2n+1)^2}\sqrt{\frac{n-1}{2}}.$$

Next, we estimate $\|\xi_0 - z\xi_0\|$. The components of $\xi_0 - z\xi_0$ for which one of $(\xi_0)_j$ and $(z\xi_0)_j$ is zero and the other is not are $\pm\frac{1}{n}, \pm\frac{2}{n}, \ldots, \pm\frac{m}{n}$, each occurring once. The sum of their squares is, using $k^2 + (m-k)^2 \leq m^2$ for $1 \leq k \leq m$, equal to

$$2\sum_{k=1}^{m}\frac{k^2}{n^2} = \frac{2m^2}{n^2} + \sum_{k=1}^{m-1}\frac{k^2+(m-k)^2}{n^2} \leq (m+1)\frac{m^2}{n^2}.$$

There are $2n-1-m$ other nonzero components, all of absolute value at most m/n, so

$$\|\xi_0 - z\xi_0\| \leq \sqrt{(2n-1-m)\frac{m^2}{n^2} + (m+1)\frac{m^2}{n^2}} = m\sqrt{\frac{2}{n}}.$$

Using $\|\xi_0\| \geq \sqrt{2n/3}$, we therefore obtain

$$\|\xi - y\xi\| \leq \frac{\pi nm}{(2n+1)^2}\sqrt{\frac{n-1}{2}}\sqrt{\frac{3}{2n}} \leq \left(\frac{\pi\sqrt{3}}{4}\right)\left(\frac{m}{2n+1}\right) = \frac{\pi\sqrt{3}}{4}\theta^{1/2}$$

and

$$\|\xi - z\xi\| \leq m\sqrt{\frac{2}{n}}\sqrt{\frac{3}{2n}} = 2\sqrt{3}\left(1+\frac{1}{2n}\right)\theta^{1/2}.$$

If n is sufficiently large, it follows that

$$\|\xi - y\xi\| \leq \tfrac{3}{2}\theta^{1/2}, \quad \|\xi - z\xi\| \leq \tfrac{7}{2}\theta^{1/2}, \quad \text{and} \quad \|y\xi - z\xi\| \leq 5\theta^{1/2}.$$

Using Lemma 6.4 on the real parts of the scalar products, and taking α, β, $\gamma \geq 0$, we calculate

$$\begin{aligned} \|(\alpha\cdot 1 + \beta y + \gamma z)\xi\|^2 &= \alpha^2 + \beta^2 + \gamma^2 + 2\alpha\beta\left(1 - \tfrac{1}{2}\|\xi - y\xi\|^2\right) \\ &\quad + 2\alpha\gamma\left(1 - \tfrac{1}{2}\|\xi - z\xi\|^2\right) + 2\beta\gamma\left(1 - \tfrac{1}{2}\|y\xi - z\xi\|^2\right) \\ &\geq (\alpha+\beta+\gamma)^2\left(1 - \tfrac{25}{2}\theta\right) \end{aligned}$$

for large n. Since $\|\xi\| = 1$, this shows that, for α, β, $\gamma \geq 0$ and n large,

$$\alpha + \beta + \gamma = 1 \quad \text{implies} \quad \|\alpha\cdot 1 + \beta y + \gamma z\| \geq \left(1 - \tfrac{25}{2}\theta\right)^{1/2}.$$

∎

Lemma 6.6. Let A and B be unital C*-algebras, with B nuclear. Let $E \subset A$ be a finite dimensional operator system, and let $E_0 \subset E$ be a subspace such that $1 \in E_0$ and $E_0 + E_0^* = E$. If $\varepsilon > 0$ and $T : E \to B$ is a unital selfadjoint linear map with $\|T|_{E_0}\|_{\mathrm{cb}} < 1 + \varepsilon$, then there exists a unital completely positive map $S : A \to B$ such that $\|S|_{E_0} - T|_{E_0}\| < \varepsilon$.

Proof: Choose $\delta > 0$ such that $\|T|_{E_0}\|_{\mathrm{cb}} < 1 + \varepsilon - \delta$. Since B is nuclear, there are n and unital completely positive maps $Q_1 : B \to M_n$ and $Q_2 : M_n \to B$ such that $\|Q_2 \circ Q_1|_{T(M)} - \mathrm{id}_B|_{T(M)}\| < \delta$. We have $\|Q_1 \circ T|_{E_0}\|_{\mathrm{cb}} < 1 + \varepsilon - \delta$, so Wittstock's generalization of the Arveson extension theorem (Theorem 7.2 of [Pl]) yields a linear map $Q_0 : E \to M_n$ such that $\|Q_0\|_{\mathrm{cb}} < 1 + \varepsilon - \delta$ and $Q_0|_{E_0} = Q_1 \circ T|_{E_0}$. Set $Q(a) = \frac{1}{2}(Q_0(a) + Q_0(a^*)^*)$. Then $Q : E \to M_n$ is linear, selfadjoint, and satisfies $\|Q\|_{\mathrm{cb}} < 1 + \varepsilon - \delta$ and $Q|_{E_0} = Q_1 \circ T|_{E_0}$. Lemma 1.9 provides a unital completely positive map $S_0 : A \to M_n$ such that $\|S_0|_E - Q\| < \varepsilon - \delta$. Then $S = Q_2 \circ S_0 : A \to B$ is a unital completely positive map such that $\|S|_{E_0} - T|_{E_0}\| < \varepsilon$. ∎

Theorem 6.7. The function ρ_0 of Notation 6.1 satisfies $\rho_0(\theta_1, \theta_2) \leq \frac{25}{4}|\theta_1 - \theta_2|$ for all $\theta_1, \theta_2 \in \mathbb{R}$.

Proof: By Lemmas 6.2 and 6.3, ρ_0 is a continuous translation invariant pseudometric. We therefore easily see that it is enough to show that for every $\varepsilon > 0$ there is $\delta > 0$ and a dense subset S of $(0, \delta)$ such that $\rho_0(\theta, 0) \leq (\frac{25}{4} + \varepsilon)\theta$ for $\theta \in S$.

Take $\varepsilon_0 = \frac{2}{25}\varepsilon$, and choose $\delta_0 > 0$ such that $(1 - r)^{-1/2} < 1 + \frac{1}{2}(1 + \varepsilon_0)r$ for $r \in (0, \delta_0)$. Take $\delta = \frac{2}{25}\delta_0$ and take

$$S = (0, \delta) \cap \left\{ \frac{m^2}{(2n+1)^2} : m, n \in \mathbb{N} \right\}.$$

Then for $\theta \in S$, Lemma 6.5 shows that the map $T_\theta : E_0(\theta) \to E_0(0)$ of that lemma satisfies

$$\|T_\theta\|_{\mathrm{cb}} \leq \left(1 - \tfrac{25}{2}\theta\right)^{-1/2} < 1 + (\tfrac{25}{4} + \varepsilon)\theta.$$

Now apply Lemma 6.6, with $E_0 = E_0(\theta)$, $E = E(\theta)$, and T defined by

$$T(1) = 1, \quad T(u(\theta)) = u(0), \quad T(u(\theta)^*) = u(0)^*, \quad T(v(\theta)) = v(0), \quad \text{and} \quad T(v(\theta)^*) = v(0)^*,$$

so that $T|_{E_0} = T_\theta$. This gives a unital completely positive map $R_0 : A(\theta) \to A(0)$ such that $\|R_0|_{E_0(\theta)} - T_\theta\| < (\frac{25}{4} + \varepsilon)\theta$. In particular, $R = \iota_0 \circ R_0 : A(\theta) \to \mathcal{O}_2$ is a unital completely positive map satisfying

$$\|R(u(\theta)) - \iota_0(u(0))\| < (\tfrac{25}{4} + \varepsilon)\theta \quad \text{and} \quad \|R(v(\theta)) - \iota_0(v(0))\| < (\tfrac{25}{4} + \varepsilon)\theta.$$

So $\rho_0(\theta, 0) \leq (\frac{25}{4} + \varepsilon)\theta$. ∎

Corollary 6.8. The field of rotation algebras, with the unitary sections defined by the standard generators, is (X_0, ρ)-embeddable in $\mathcal{O}_2$ for a one point space X_0 and with $\rho(r) = 28r^{1/2}$.

Proof: This follows from Remark 4.16 and the inequality $11\sqrt{25/4} < 28$. ∎

Theorem 6.9. There exists a unital injective representation φ of the continuous field of rotation algebras, regarded as defined over $\mathbb{R}$, which is periodic in the sense that $\varphi_{\theta+1} = \varphi_\theta$ for all $\theta \in \mathbb{R}$, and for which there is a constant C such that for all $\theta_1, \theta_2 \in \mathbb{R}$,

$$\max(\|\varphi_{\theta_1}(u(\theta_1)) - \varphi_{\theta_2}(u(\theta_2))\|, \|\varphi_{\theta_1}(v(\theta_1)) - \varphi_{\theta_2}(v(\theta_2))\|) < C|\theta_1 - \theta_2|^{1/2}.$$

Moreover, C can be chosen smaller than $840{,}000 = 8.4 \cdot 10^5$.

Proof: We describe the changes that must be made to the proofs of Lemma 5.6 and Theorem 5.10. Take the spaces X and Y of Lemma 5.6 to be one point spaces (as in the proof of Theorem 5.10), and use $\mathbb{R}$ in place of $[0, 1]$. Choose a single injective unital homomorphism $\alpha : A(0) \to \mathcal{O}_2$, and take $\varphi_n = \alpha$ for every integer n. We carry out the rest of the construction on $[0, 1]$, repeating each step using periodicity in each $[n, n+1]$. (The only reason for not restricting to $[0, 1]$ at the beginning is to ensure that we obtain the distance estimate of the theorem for, say, $\theta_1 < 1 < \theta_2$.)

We will take all sectional distances $d_{\mathrm{S}}(\varphi_{\theta_1}, \varphi_{\theta_2})$ with respect to the continuous sections u and v. By Theorem 6.7, we have $\rho_0(\theta_1, \theta_2) < \frac{25}{4}|\theta_1 - \theta_2|$. So, in the proof of Theorem 5.10, we take $\alpha = 1$. We follow the modification described for this value of α in Remark 5.11.

Construct injective unital homomorphisms $\varphi_\theta : A(\theta) \to \mathcal{O}_2$ for θ in the set

$$S_0 = \{j/n^k : k \geq 0, \ 0 \leq j \leq n^k\}$$

as in the proof of Lemma 5.6, with $n_1, n_2, \ldots,$ and n' all equal to 90^2 (see Remark 5.11), and extend over $S = \{j/n^k : k \geq 0, \ j \in \mathbb{Z}\}$ by periodicity. As in Remark 5.11, if $\theta_1, \theta_2 \in S$ then

$$d_{\mathrm{S}}(\varphi_{\theta_1}, \varphi_{\theta_2}) \leq 330,000 \left(\tfrac{25}{4}\right)^{1/2} |\theta_1 - \theta_2|^{1/2} \leq 840,000|\theta_1 - \theta_2|^{1/2}.$$

This is true for all $\theta_1, \theta_2 \in S$, but clearly extends by continuity to all $\theta_1, \theta_2 \in \mathbb{R}$. ∎

Corollary 6.10. There exist continuous functions $u, v : S^1 \to U(\mathcal{O}_2)$ such that:

(1) $u(\zeta)v(\zeta) = \zeta v(\zeta)u(\zeta)$ for all $\zeta \in S^1$.

(2) $C^*(u(\zeta), v(\zeta))$ is isomorphic to the universal C*-algebra on unitaries u and v satisfying $uv = \zeta vu$.

(3) There is a constant C such that for all $\zeta_1, \zeta_2 \in S^1$, we have

$$\|u(\zeta_1) - u(\zeta_2)\| \leq C|\zeta_1 - \zeta_2|^{1/2} \quad \text{and} \quad \|v(\zeta_1) - v(\zeta_2)\| \leq C|\zeta_1 - \zeta_2|^{1/2}.$$

Moreover, C can be chosen less than $420,000$.

Proof: This follows from the theorem, because if $\zeta_1, \zeta_2 \in S^1$, then there exist $\theta_1, \theta_2 \in \mathbb{R}$ such that

$$\exp(2\pi i\theta_1) = \zeta_1, \quad \exp(2\pi i\theta_2) = \zeta_2, \quad \text{and} \quad |\theta_1 - \theta_2| \leq \tfrac{1}{4}|\zeta_1 - \zeta_2|.$$

It follows that $|\theta_1 - \theta_2|^{1/2} \leq \frac{1}{2}|\zeta_1 - \zeta_2|^{1/2}$. ∎

Remark 6.11. No better exponent is possible in Theorem 6.9 or in Corollary 6.10, because Proposition 4.6 of [HR] shows no better exponent is possible even for representations on a Hilbert space. This implies that the square root in Proposition 4.13 can't be removed, and also provides a (rather indirect) proof that the exponent $\frac{1}{2}$ in Lemma 1.12 can't be improved. Moreover, in the construction of embeddings in $\mathcal{O}_2$, the exponent $\frac{1}{2}$ in Lemma 1.12 can't be evaded by using some other proof.

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